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Some Variants of Ĉech Δ −Normal Closure Space

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ABSTRACT : **This manuscript is devoted to study several generalized notions of Ĉech normality in Ĉech Closure space. This work examines different kinds of Ĉech normality that are located between the Ĉech Δ −normal and Ĉech κ −normal spaces. The relations between these distinctions of Ĉech normality are established, and some of examples are provided to support it. By using Ĉech β −normal space, novel normality decompositions are obtained.**

Keywords: Ĉech closure space; Ĉech δ − closed set; Ĉech π −closed set; Ĉech Δ −normal (Ĉech weakly Δ −normal) spaces.

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1 Introduction

The literature reveals that topological structures more general than classical topology are better suited for studying digital topology, network theory, image processing, and related fields. Different generalized structures, such as closure spaces, generalized closure spaces closure, Ĉech closure spaces, generalized topologies (GT), weak structures (WS), and minimal spaces, have been defined and considered (see [1, 3, 4, 5, 7, 13, 15]).

Newly, Ĉech closure spaces have garnered significant attention due to their applicability in these fields. The utility of Ĉech closure spaces in digital topology, computer graphics, image processing, and pattern recognition is well-documented [12, 20, 21, 24]. Ĉech closure spaces, defined by Ĉech [2], are derived from Kuratowski closure operators by omitting the idempotent condition.

The concept of a normal topological space has long intrigued topologists [6, 8, 23] and holds a prominent position in general topology. Unlike other separation axioms, normality exhibits unique behavior concerning subspaces.

A lot of generalized notions of normality have been introduced, including quasi normality [10], almost normality [25], mildly normal (κ −normal) [23, 25], pre −normal [17] (p −normal [18]), and β − normal [19]. Additional variations include Δ −normal, weakly Δ −normal, and weakly functionally Δ −normal spaces, presented by Das in 2009 [6]. Other related concepts are π β −normal [22], almost p −normal, mildly p −normal [15]. As a generalization of normal, β −normal, and almost normal spaces [1, 26], Das et al. recently introduced almost β −normal spaces [9].

This paper examines Ĉech normality variations that are located between the Δ −normal and κ −normal spaces. The interrelationship of these distinctions of Ĉech normality is established, and examples have been provided to support it. New normality decompositions are obtained using Ĉech β −normal spaces.

2 Preliminaries

In this section, a number of the meaningful definitions and outcomes will be reviewed.

Definition 2.1 [1] Suppose that an operator $\hat{c}l: P(\Gamma) \rightarrow P(\Gamma)$ (where $P(\Gamma)$ denotes the power set of Γ), and

1. $\hat{c}l(\phi) = \phi$.

2. $\Omega \subseteq \hat{c}l(\Omega), \forall \Omega \subseteq \Gamma$ (enlarging).

3. $\hat{c}l(\Omega \cup \psi) = \hat{c}l(\Omega) \cup \hat{c}l(\psi), \forall\ \Omega, \psi \subseteq \Gamma$ (additive).

4. $\hat{c}l(\hat{c}l(\Omega) = \hat{c}l(\Omega), \forall\ \Omega \subseteq \Gamma$ (idempotent).

If $\hat{c}l$ satisfies the axioms 1, 2, 3, then this structure is called Ĉech closure operator and the pair $(\Gamma, \hat{c}l)$ is called a Ĉech closure space. The definition of a topological space can be reconstructed, if the Ĉech closure operator $\hat{c}l$ is idempotent i.e satisfies the condition 4, and so Ĉech closure space is called closure space.

A subset Ω is closed with respect to Ĉech closure space $(\Gamma, \hat{c}l)$ i.e Ω is Ĉech closed, if $\hat{c}l(\Omega) = \Omega$ and it is Ĉech open if its complement is Ĉech closed i.e, if $\hat{c}l(\Omega^c) = \Omega^c$.

The empty set ϕ and the whole space Γ are both Ĉech open and Ĉech closed. The set $\hat{i}nt(\Omega)$ with respect to the Ĉech closure operator $\hat{c}l$ is appointed as $\hat{i}nt(\Omega) = \Gamma - \hat{c}l(\Gamma - \Omega)$.

Definition 2.2 [19]A subset Ω of a $\hat{C}$ech closure space $(\Gamma, \hat{c}l)$ is said to be

1. Ĉech regular open if $\Omega = \hat{i}nt(\hat{c}l(\Omega))$;

2. Ĉech regular closed if $\Omega = \hat{c}l(\hat{i}nt(\Omega))$;

3. Ĉech π −closed if it i s equal to the intersection of two Ĉech regularly closed sets.

4. Ĉech δ −open if for each $x \in \Omega$, there exists a Ĉech regular open set G such that $x \in G \subset \Omega$;

5. Ĉech $\delta -$ closed if the complement of Ω is Ĉech $\delta -$ open;

The smallest Ĉech $\delta -$ closed set containing Ω is called Ĉech $\delta -$ closure of Ω and it is denoted by $\hat{c}l_\delta(\Omega)$ and the largest Ĉech $\delta -$ open set contained in Ω is called Ĉech $\delta -$ interior of Ω and it is denoted by $\hat{i}nt_\delta(\Omega)$.

Definition 2.3 [11]A $\hat{C}$ech closure space $(\Gamma, \hat{c}l)$ is said to be

1. Ĉech normal if for every two disjoint Ĉech closed sets $\Omega = \hat{c}l(\Omega)$ and $\psi = \hat{c}l(\psi)$, there exist disjoint Ĉech open sets U and V containing $\hat{c}l(\Omega)$ and $\hat{c}l(\psi)$ respectively;

2. Ĉech $\beta -$ normal if for two disjoint Ĉech closed sets $\hat{c}l(\Omega) = \Omega$ and $\hat{c}l(\psi) = \psi$, there exist disjoint Ĉech open sets U and V such that $\hat{c}l(\Omega \cap U) = (\Omega)$, $\hat{c}l(\psi \cap V) = \hat{c}l(\psi)$ and $\hat{c}l(U) \cap \hat{c}l(V) = \phi$;

3. Ĉech almost normal if for every two disjoint Ĉech closed sets $\hat{c}l(\Omega) = \Omega$ and $\hat{c}l(\psi) = \psi$ out of which one is regularly closed, there exist disjoint Ĉech open sets U and V containing $\hat{c}l(\Omega)$ and $\hat{c}l(\psi)$ respectively;

4. Ĉech $\kappa -$ normal if for two disjoint Ĉech regularly closed sets Ω and ψ, there exist disjoint Ĉech open sets U and V containing Ω and ψ respectively;

5. Ĉech π −normal if for every two disjoint Ĉech closed sets one of which is Ĉech π −closed, there exist two disjoint Ĉech open sets U and V containing the Ĉech closed set and the Ĉech π −closed set respectively;

6. Ĉech quasi normal if for two disjoint Ĉech π −closed sets, there exist disjoint Ĉech open sets separating them;

7. Ĉech regular if for a Ĉech closed set $\hat{c}l(\Omega) = \Omega$ and a point $x \notin \hat{c}l(\Omega)$ there exist disjoint Ĉech open sets U and V such that $x \in U$ and $\hat{c}l(\Omega) \subseteq V$.

Definition 2.4 [1]A $\hat{C}$ech closure space $(\Gamma, \hat{c}l)$ is said to be

1. Ĉech T_1 if for two distinct points x and y, we have $x \notin \hat{c}l(\{y\})$ and $y \notin \hat{c}l(\{x\})$;

2. Ĉech T_2 if any two distinct points x and y are separated by disjoint Ĉech open sets. In other words, for any two distinct points x and y in the space, there exist disjoint Ĉech open sets U and V such that $x \in U$ and $y \in V$.

Lemma 2.5 [1] If U and V are subsets of a closure space $(\Gamma, \hat{c}l)$ such that $U \subseteq V$ then $\hat{c}l(U) \subseteq \hat{c}l(V)$.

3. Some Variants of Ĉech $\triangle$ −Normal Closure Space

In this part, the notions of Ĉech $\triangle -$ normal closure, and Ĉech weakly $\triangle$ −normal spaces will be presented and some of their basic properties will be studied.

Definition 3.1 A $\hat{C}$ech closure space $(\Gamma, \hat{c}l)$ is said to be

1. Ĉech $\triangle$ −normal if every pair of disjoint Ĉech closed sets one of which is Ĉech δ −closed are contained in disjoint Ĉech open sets;

2. Ĉech weakly $\triangle$ −normal if every pair of disjoint Ĉech δ −closed sets are contained in disjoint Ĉech open sets.

Example 3.2 Let $\Gamma = \{a, b, c\}$. Define $\hat{C}$ech closure operator as follows: $\hat{c}l(\phi) = \phi$, $\hat{c}l(\Gamma) = \hat{c}l(\{a, c\}) = \hat{c}l(\{a, b\}) = \Gamma$, $\hat{c}l(\{a\}) = \{a, b\}$, $\hat{c}l(\{b, c\}) = \hat{c}l(\{b\}) = \{ b, c\}$, $\hat{c}l(\{c\}) = \{c\}$.

It is clear that the Ĉech open sets are $\{\{a\}, \{a, b\}, \Gamma, \phi\}$.

The Ĉech $\delta -$ open sets are $\{\Gamma, \phi\}$.

The Ĉech $\delta -$ closed sets are $\{\Gamma, \phi\}$.

Then the Ĉech closure space $(\Gamma, \hat{c}l)$ is Ĉech weakly $\triangle$ −normal and Ĉech $\triangle$ −normal.

Theorem 3.3 If a $\hat{C}$ech closure space $(\Gamma, \hat{c}l)$ is a $\hat{C}$ech $\triangle$ −normal then for every $\hat{C}$ech closed set Ω and every $\hat{C}$ech δ −open set U containing Ω there exists an $\hat{C}$ech open set V such that $\hat{c}l(\Omega) \subseteq V \subseteq \hat{c}l(V) \subseteq U$.

Proof. Let $\hat{c}l(\Omega) = \Omega$ be a Ĉech closed set and U be an Ĉech δ −open set containing $\hat{c}l(\Omega)$. Since, $(\Gamma, \hat{c}l)$ is Ĉech $\triangle$ −normal, there exist disjoint Ĉech open sets V and W such that $\hat{c}l(\Omega) \subseteq V$ and $(\Gamma - U) \subseteq W$ implies $V \subseteq (\Gamma - W)$. Thus, by Lemma 2.5, $\hat{c}l(V) \subseteq \hat{c}l(\Gamma - W)$ implies $W \subseteq \Gamma - \hat{c}l(V)$. Therefore, $(\Gamma - U) \subseteq W \subseteq \Gamma - \hat{c}l(V)$ and hence $\hat{c}l(\Omega) \subseteq V \subseteq \hat{c}l(V) \subseteq U$.

Theorem 3.4 A $\hat{C}$ech closure space $(\Gamma, \hat{c}l)$ is $\hat{C}$ech weakly $\triangle$ −normal if and only if for every $\hat{C}$ech δ −closed set Ω and a $\hat{C}$ech δ −open set U containing Ω there is an $\hat{C}$ech open set V such that $\hat{c}l(\Omega) \subset V \subset \hat{c}l(V) \subset U$.

Proof. Let $\hat{c}l(\Omega) = \Omega$ be aĈech closed set and U be a Ĉech δ −open set containing $\hat{c}l(\Omega)$ implies $(\Gamma - U)$ is a Ĉech δ −closed set which is disjoint from the Ĉech closed set Ω. Since, $(\Gamma, \hat{c}l)$ Ĉech weakly $\triangle$ −normal, there exist disjoint Ĉech open sets V and W such that $\hat{c}l(\Omega) \subseteq V$ and $(\Gamma - U) \subseteq W$. Thus, $V \subseteq (\Gamma - W)$ implies $\hat{c}l(V) \subseteq \hat{c}l(\Gamma - W) = (\Gamma - W)$, and so, $W \subseteq (\Gamma - \hat{c}l(V))$. Therefore, $(\Gamma - U) \subseteq W \subseteq (\Gamma - \hat{c}l(V))$ and hence $\hat{c}l(\Omega) \subseteq V \subseteq \hat{c}l(V) \subseteq U$.

Conversely, let Ω and ψ be two disjoint Ĉech δ −closed sets in Γ and $\hat{c}l(\Omega) = \Omega$. Then $U = \Gamma - \psi$ is a Ĉech δ −open set containing the Ĉech δ −closed set $\hat{c}l(\Omega)$. Thus, by the hypothesis there exists an Ĉech open set V such that $\hat{c}l(\Omega) \subset V \subset \hat{c}l(V) \subset U$. Then V and $\Gamma - \hat{c}l(V)$ are disjoint Čech open sets containing Ω and ψ, respectively. Hence $(\Gamma, \hat{c}l)$ is Čech weakly $\triangle$ −normal.

Corollary 3.5 Theorem (3.3) deals with $\hat{C}$ech closed sets and $\hat{C}$ech δ −open sets, but theorem (3.4) involves $\hat{C}$ech δ −closed sets and $\hat{C}$ech δ −open sets. This is a stricter requirement since δ −closed sets are a subset of $\hat{C}$ech closed sets.

Some lemmas will be studied, through which some following theories will be proven.

Lemma 3.6 In a $\hat{C}$ech closure space $(\Gamma, \hat{c}l)$, every $\hat{C}$ech π −closed set is $\hat{C}$ech δ − closed.

Proof. Let $x \in \Omega^c$ i.e $x \notin \Omega$. Since Ω is Čech π −closed set, then there exist two Čech regularly closed sets ψ, E such that $\Omega = \psi \cap E$. Hence $x \notin \psi$ or $x \notin E$. Suppose that $x \notin \psi$.Then $x \in \psi^c$ and ψ^c is Čech regular open with $\psi^c \subseteq \Omega^c$. Therefore Ω^c is Čech δ −open set and so Ω is Čech δ −closed.

Lemma 3.7 In a $\hat{C}$ech closure space $(\Gamma, \hat{c}l)$, every $\hat{C}$ech δ −closed set is $\hat{C}$ech closed.

Proof. Suppose that Ω is Čech δ −closed set. We shall prove that $\hat{c}l(\Omega) \subseteq \Omega$. Let $x \notin \Omega$ i.e $\in \Omega^c$. Since Ω^c is Čech δ −open set, then there exists Čech regular open set G such that $x \in G \subseteq \Omega^c$. Hence $G \cap \Omega = \phi$. Therefore, $x \notin \hat{c}l(\Omega)$ and so $\hat{c}l(\Omega) = \Omega$ i.e Ω is Čech closed.

The following example shows that not every Čech δ − closed set is necessarily a Čech π −closed.

Example 3.8 Let $\Gamma = \{a, b, c\}$. Define $\hat{C}$ech closure operator as follows:

$\hat{c}l(\phi) = \phi$, $\hat{c}l(\Gamma) = \hat{c}l(\{a,\ c\}) = \hat{c}l(\{a,\ b\}) = \Gamma$, $\hat{c}l(\{a\}) = \{a,\ b\}$, $\hat{c}l(\{b,\ c\}) = \hat{c}l(\{b\}) = \{\ b,\ c\}$, $\hat{c}l(\{c\}) = \{b, c\}$. It is clear that:

The Čech open sets are $\{\{a\}, \Gamma, \phi\}$.

The Čech closed sets are $\{\{b, c\}, \Gamma, \phi\}$.

The Čech δ − closed sets are $\{\{b\}, \{a, b\}, \{b, c\}, \Gamma, \phi\}$.

The Čech π − closed sets are $\{\{b, c\}, \Gamma, \phi\}$.

Then the set $\{a, b\}$ is a Čech δ −closed set, but not a Čech π −closed.

The following example shows that not every Čech closed set is necessarily a Čech δ −closed.

Example 3.9 Let $\Gamma = \{a, b, c, d\}$. Define $\hat{C}$ech closure operator as follows: $\hat{c}l(\{a\}) = \{a\}$, $\hat{c}l(\{b\}) = \hat{c}l(\{a, b\}) = \{a, b\}$, $\hat{c}l(\{c\}) = \hat{c}l(\{c, d\}) = \{b, c, d\}$, $\hat{c}l(\{a, d\}) = \{a, d\}$, $\hat{c}l(\{d\}) = \{d\}$, $\hat{c}l(\{b, d\}) = \hat{c}l(\{a, b, d\}) = \{a, b, d\}$, $\hat{c}l(\Gamma) = \hat{c}l(\{a, c\}) = \hat{c}l(\{a, c, d\}) = \hat{c}l(\{a, b, c\}) = \hat{c}l(\{b, c\}) = \hat{c}l(\{b, c, d\}) = \Gamma$, $\hat{c}l(\phi) = \phi$. It is clear that:

The Čech closed sets are $\{\{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, d\}, \Gamma, \phi\}$.

The Čech open sets are $\{\{b, c, d\}, \{a, b, c\}, \{c, d\}, \{b, c\}, \{c\}, \Gamma, \phi\}$.

The Čech δ − closed are $\{\Gamma, \phi\}$.

Then the set $\{d\}$ is a Čech closed set, but not a Čech δ −closed.

Remark 3.10 The lemmas (see 3.6 and 3.7) immediately lead to the following implications: 2, 3 However, none of these implications in Figure 1 are reversible. Refer to Examples 3.8 and 3.9.

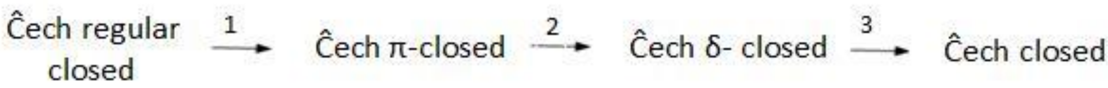

Figure 1

The following theorems offer several decompositions of normality in terms of intermediate variations of Ĉech normality, which fall between Ĉech normal and Ĉech κ −normal spaces.

Theorem 3.11 In a $\hat{C}$ech closure space $(\Gamma, \hat{c}l)$, every $\hat{C}$ech normal space is $\hat{C}$ech Δ −normal.

Proof. Let Ω be Ĉech δ −closed, and ψ be Ĉech closed with $\Omega \cap \psi = \phi$. By Lemma (3.7) A is Ĉech closed. Since $(\Gamma, \hat{c}l)$ is Ĉech normal space, then there exist disjoint Ĉech open sets U, V such that $\Omega \subseteq U$, $\psi \subseteq V$. Hence $(\Gamma, \hat{c}l)$ is Ĉech Δ −normal space.

Theorem 3.12 In a $\hat{C}$ech closure space $(\Gamma, \hat{c}l)$, every $\hat{C}$ech Δ −normal space is $\hat{C}$ech π −normal.

Proof. Let Ω be Ĉech π −closed, and ψ be Ĉech closed with $\Omega \cap \psi = \phi$. By Lemma (3.6) A is Ĉech δ −closed. Since $(\Gamma, \hat{c}l)$ is Ĉech Δ −normal space, then there exist disjoint Ĉech open sets U, V such that $\Omega \subseteq U, \psi \subseteq V$. Hence $(\Gamma, \hat{c}l)$ is Ĉech π −normal space.

Theorem 3.13 In a $\hat{C}$ech closure space $(\Gamma, \hat{c}l)$, every $\hat{C}$ech Δ −normal space is $\hat{C}$ech weakly Δ −normal.

Proof. Let Ω and ψ be two disjoint Ĉech δ −closed sets in $(\Gamma, \hat{c}l)$. By Lemma (3.7) A is Ĉech closed. Since $(\Gamma, \hat{c}l)$ is Ĉech Δ −normal space, then there exist disjoint Ĉech open sets U, V such that $\Omega \subseteq U, \psi \subseteq V$. Hence $(\Gamma, \hat{c}l)$ is Ĉech weakly Δ −normal space.

Theorem 3.14 In a $\hat{C}$ech closure space $(\Gamma, \hat{c}l)$, every $\hat{C}$ech weakly Δ −normal space is $\hat{C}$ech quasi normal.

Proof. Let Ω and ψ be two disjoint Ĉech π −closed sets in $(\Gamma, \hat{c}l)$. By Lemma (3.6) A is Ĉech δ −closed. Since $(\Gamma, \hat{c}l)$ is Ĉech weakly Δ −normal space, then there exist disjoint Ĉech open sets U, V such that $\Omega \subseteq U, \psi \subseteq V$. Hence $(\Gamma, \hat{c}l)$ is Ĉech quasi normal space.

An idempotent Ĉech closure space which is Ĉech $\triangle$ −normal but not Ĉech normal.

Example 3.15 Let $\Gamma = \{a, b, c, d\}$. Define idempotent $\hat{C}$ech closure operator as follows:

$\hat{c}l(\{a\}) = \{a\}$, $\hat{c}l(\{b\}) = \hat{c}l(\{a,\ b\}) = \hat{c}l(\{a,\ b,\ c\}) = \{a,\ b,\ c\}$, $\hat{c}l(\{c\}) = \{c\}$, $\hat{c}l(\{a,\ c\}) = \{a, c\}$, $\hat{c}l(\{c,\ d\}) = \{c, d\})$, $\hat{c}l(\{d\}) = \{d\}$, $\hat{c}l(\{a,\ d\}) = \{a,\ d\}$, $\hat{c}l(\{b,\ c\}) = \{a,\ b,\ c\}$, $\hat{c}l(\{a,\ c,\ d\}) = \hat{c}l(\Gamma) = \hat{c}l(\{b, d\}) = \hat{c}l(\{a, b, d\}) = \hat{c}l(\{b, c, d\}) = \Gamma$, $\hat{c}l(\phi) = \phi$. Then we find that:

The Ĉech closed sets are $\{\{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, b, c\}, \Gamma, \phi\}$.

The Ĉech open sets are $\{\{b, c, d\}, \{a, b, d\}, \{a, b, c\}, \{b, d\}, \{b, c\}, \{a, b\}, \{d\}, \Gamma, \phi\}$.

The Ĉech δ − closed are $\{\{a, b, c\}, \{d\}, \Gamma, \phi\}$.

Clearly, $(\Gamma, \hat{c}l)$ is a idempotent Ĉech closure space which is vacuously Ĉech $\triangle$ −normal but not Ĉech normal because for the two disjoint Ĉech closed sets $\{a\}, \{c\}$,there does not exist disjoint Ĉech open sets containing them respectively

An idempotent Ĉech closure space which is Ĉech weakly $\triangle$ −normal but not Ĉech $\triangle$ −normal.

Example 3.16 Let $\Gamma = \{a, b, c, d\}$. Define idempotent $\hat{C}$ech closure operator as follows:

$\hat{c}l(\{a\}) = \{a\}$, $\hat{c}l(\{b\}) = \hat{c}l(\{a, b\}) = \{a, b\}$, $\hat{c}l(\{c\}) = \hat{c}l(\{a, c\}) = \hat{c}l(\{c, d\}) = \hat{c}l(\{a, c, d\}) = \{a, c, d\}$, $\hat{c}l(\{d\}) = \{d\}$, $\hat{c}l(\{a, d\}) = \{a, d\}$, $\hat{c}l(\{b, c\}) = \hat{c}l(\{a, b, c\}) = \hat{c}l(\{b, c, d\}) = \hat{c}l(\Gamma) = \Gamma$, $\hat{c}l(\{b, d\}) = \hat{c}l(\{a, b, d\}) = \{a, b, d\}$, $\hat{c}l(\phi) = \phi$.Then We find that:

The Čech closed sets are $\{\{a\}, \{a, b\}, \{a, c, d\}, \{d\}, \{a, d\}, \{a, b, d\}, \Gamma, \phi\}$

The Čech open sets are $\{\{b, c, d\}, \{c, d\}, \{b\}, \{a, b, c\}, \{b, c\}, \{c\}, \Gamma, \phi\}$

The Čech $\delta -$ closed are $\{\{a\}, \{a, b\}, \{a, c, d\}, \Gamma, \phi\}$.

This space is Čech weakly $\triangle -$normal but not Čech $\triangle -$normal. Because for the Čech $\delta -$ closed set $\{a\}$ and a Čech closed set $\{d\}$, there are no disjoint Čech open sets containing $\{a\}$ and $\{d\}$.

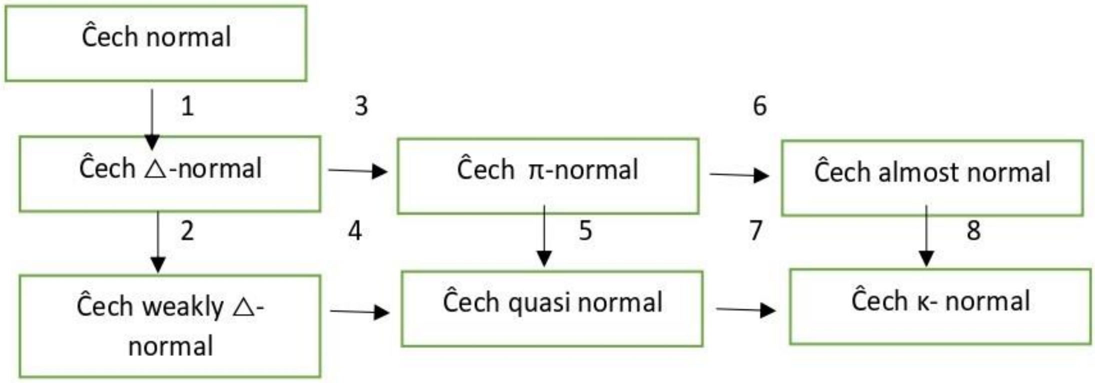

Figure 2

Remark 3.17 The theorems (refer to 3.11–3.14) directly result in the following implications: The sequence consists of the numbers 1, 2, 3, and 4. Nevertheless, none of the consequences depicted in Figure 2 can be undone. Consult the provided examples (refer to 3.15–3.16).

Through our analysis, we have examined the correlation among several alternative forms of Čech $\beta -$normality and have determined that they have all been interconnected.

Theorem 3.18 If a $\hat{C}$ech closure space $(\Gamma, \hat{c}l)$ is $\hat{C}$ech $\beta -$normal and satisfies the $\hat{C}$ech T_1, then $(\Gamma, \hat{c}l)$ is a $\hat{C}$ech regular space.

Proof. Suppose that $(\Gamma, \hat{c}l)$ is Čech $\beta -$normal, and let $\Omega = cl(\Omega) \subseteq \Gamma$ be a Čech closed set and $x \in \Gamma$ such that $\notin cl(\Omega)$. Since $cl(\Omega)$ and $\{x\}$ are disjoint Čech closed sets, where $\{x\}$ is also Čech closed due to the Čech T_1. By Čech $\beta -$normality, there exist disjoint Čech open sets U and V such that $cl(\Omega \cap U) = \Omega$ $(i.e.\ cl(\Omega) \subseteq cl(U))$, $cl(\{x\} \cap V) = \{x\}$ $(i.e.\ \{x\} \subseteq cl(V))$ and $cl(U) \cap cl(V) = \phi$. Since $x \in V$ implies $x \notin cl(U)$, $cl(\Omega) \subseteq U$, and $U \cap V = \phi$, therefore $(\Gamma, \hat{c}l)$ is Čech regular.

Theorem 3.19 A $\hat{C}$ech closure space $(\Gamma, \hat{c}l)$ is $\hat{C}$ech $\beta -$normal if, for each $\hat{C}$ech closed set $\hat{c}l(\Omega) \subseteq \Gamma$ and for all open $U \subseteq \Gamma$ with $\Omega \subseteq U$, there exists an $\hat{C}$ech open set $V \subseteq \Gamma$ such that $\hat{c}l(V \cap \Omega) = \Omega$ and $\Omega \subseteq \hat{c}l(V) \subseteq U$.

Proof. Suppose that $(\Gamma, \hat{c}l)$ is Čech $\beta -$normal, and let $\hat{c}l(\Omega)$ be a Čech closed set and U be an Čech open set such that $\Omega \subseteq U$. Then $\hat{c}l(\Omega)$ and $\Gamma - U$ are disjoint Čech closed sets, and by Čech $\beta -$normality, there exist disjoint Čech open sets V and W such that $\hat{c}l(V) \cap \hat{c}l(W) = \phi, \hat{c}l(V \cap \Omega) = \Omega$, and $\hat{c}l(W \cap (\Gamma - U)) = \Gamma - U$. Since V is Čech open and Ω is Čech closed such that $\Omega \subseteq U$. Hence, $\Omega \subseteq cl(V)$ and $cl(V) \subseteq \Gamma - cl(W) \subseteq U$. Since $\Gamma - U \subseteq cl(W)$, hence $\Omega \subseteq cl(V) \subseteq U$.

Theorem 3.20 In a $\hat{C}$ech $\beta -$normal space the following statements are equivalent:

1. $(\Gamma, \hat{c}l)$ is Ĉech normal.
2. $(\Gamma, \hat{c}l)$ is Ĉech Δ −normal.
3. $(\Gamma, \hat{c}l)$ is Ĉech weakly Δ −normal.
4. $(\Gamma, \hat{c}l)$ is Ĉech quasi normal.
5. $(\Gamma, \hat{c}l)$ is Ĉech κ −normal.

Proof. Since $(\Gamma, \hat{c}l)$ is Ĉech β −normal, for disjoint Ĉech closed sets $D = cl(D)$ and $E = cl(E)$, we already have the Ĉech open sets M and N such that $\hat{c}l(M) \cap \hat{c}l(N) = \phi$, $\hat{c}l(D \cap M) = D$ and $\hat{c}l(E \cap N) = E$. This implies $D \subseteq M$ and $E \subseteq N$, fulfilling the conditions for Čech normality.

$1 \rightarrow 2$ Let D be Ĉech δ −closed, and E be Ĉech closed with $D \cap E = \phi$. By Lemma (3.4), then D is Ĉech closed. Since $(\Gamma, \hat{c}l)$ is Ĉech normal space, then there exist disjoint Ĉech open sets M, N such that $D \subseteq M$ and $E \subseteq N$. Hence $(\Gamma, \hat{c}l)$ is Ĉech Δ −normal space.

$2 \rightarrow 3$ Let D and E be two disjoint Ĉech δ −closed sets in $(\Gamma, \hat{c}l)$. By Lemma (3.4), then D is Ĉech closed. Since $(\Gamma, \hat{c}l)$ is Ĉech Δ −normal space, then there exist disjoint Ĉech open sets M, N such that $D \subseteq M$ and $E \subseteq N$. Hence $(\Gamma, \hat{c}l)$ is Ĉech weakly Δ −normal

$3 \rightarrow 4$ Let D and E be two disjoint Ĉech π −closed sets in $(\Gamma, \hat{c}l)$. By Lemma (3.3) D is Ĉech δ −closed. Since $(\Gamma, \hat{c}l)$ is Ĉech weakly Δ −normal space, then there exist disjoint Ĉech open sets M, N such that $D \subseteq M$ and $E \subseteq N$. Hence $(\Gamma, \hat{c}l)$ is Ĉech quasi normal space.

$4 \rightarrow 5$ Let D and E be two disjoint Ĉech regularly closed sets. From figure (1), every Ĉech regularly closed set is Ĉech π −closed. Since $(\Gamma, \hat{c}l)$ is Čech quasi normal, then there exist disjoint Ĉech open sets M, N such that $D \subseteq M$ and $E \subseteq N$. Hence $(\Gamma, \hat{c}l)$ is Ĉech κ −normal.

$5 \rightarrow 1$ Let D and E be two disjoint Ĉech closed sets. Similarity every Ĉech closed set is Ĉech regularly closed. Since $(\Gamma, \hat{c}l)$ is Ĉech κ −normal, then there exist disjoint Ĉech open sets M, N such that $D \subseteq M$ and $E \subseteq N$. Hence $(\Gamma, \hat{c}l)$ is Ĉech normal.

Theorem 3.21 In a *Ĉech* β −normal space the following statements are equivalent:

1. $(\Gamma, \hat{c}l)$ is Ĉech normal.
2. $(\Gamma, \hat{c}l)$ is Ĉech Δ −normal.
3. $(\Gamma, \hat{c}l)$ is Ĉech π −normal.
4. $(\Gamma, \hat{c}l)$ is Ĉech almost normal.
5. $(\Gamma, \hat{c}l)$ is Ĉech κ −normal.

Proof. The prove of the implication $1 \rightarrow 5$ is similar to the prove of theorem (3.20).

4 Conclusion

In the framework of Ĉech closure spaces, It has been investigated variations of Ĉech normality situated between Ĉech Δ −normal and Ĉech κ −normal spaces. The correlation between these several aspects of Ĉech normality has been proven, and illustrative instances have been offered to substantiate it. Ĉech β −normal spaces are utilized to produce decompositions of the new normality.

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