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Article Review on Alpha Power Transformation Generalized Pareto Distribution

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ABSTRACT **: In this article some statistical properties of APTGP distribution are addressed such as moments, arithmetic mean, moment generating function, quantile function, reliability, and hazard function. Also will focus on compared the proposed distribution with other forms of Pareto distribution, such as Pareto distribution, alpha-power Pareto distribution, and generalized Pareto distribution. One real datasets have been considered to compare the proposed distribution with other forms of Pareto distribution and to show the usefulness of the proposed distribution. The comparison has been shown that the proposed distribution is more flexible than the others.**

KEYWORDS: α-Power Transformation Method, alpha-power Pareto, Generalized Pareto, Moments, Pareto, Reliability Function, Hazard Function.

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I. INTRODUCTION

Pareto distribution is a probability distribution that has many applications, and is widely used in applications of reliability theory because it is one of the failure distributions of stress models in mechanical engineering. It's successfully used by Philbrick (1985) for projection of losses in an insurance company, real state and liability experience of hospitals Farshchian (2010). It's was applied to model sea clutter intensity returns, Levy (2003) used Pareto distribution for investigation of wealth in society.

Van Montfort and Witter (1986) discussed the fitting of the generalized Pareto distribution to Dutch peaks-over-threshold (POT) rainfall series. They used Maximum likelihood (ML) estimation of the parameters and discussed the improvement of the fit in the right tail by left censoring of POT series.

Castillo (1997) considered generalized form of Pareto distribution to model exceedances over a margin in flood control. The Power Transformed Model (PTM) was discussed by Gupta and Kundu (2009) as a parameter induction method. Another parameter induction method is that introduced by Marshall and Olkin (1997). Gupta and Kundu (2009) identified the generalized distributions derived from Marshall and Olkin parameter induction method as proportional odds models. The alpha power transformed method was introduced by Mahdavi and Kundu (2017).

Analysis of life time data depends on the knowledge of the distributions of that data, and sometimes the data sets that follow classic distributions are exceptional and even not realistic. From this point, a new class of Pareto distribution will be introduced in this paper, through development it to another distribution adopting the Alpha Power Transformation (APT) Method which proposed by Mahdavi and Kundu (2017). It's called α-power transformation generalized Pareto distribution. Some statistical properties of the proposed distribution are being studied, such as moments, arithmetic mean, moment generating function, random variables, reliability, and hazard function. Also will focus on compared the proposed distribution with other forms of Pareto distribution, such as Pareto distribution, alpha-power Pareto distribution, and generalized Pareto distribution by applying them to real data.

2. SOME KINDS OF PARETO DISTRIBUTION

The Pareto distribution of 1^{st} kind is defined by Johnson and Balakrishnan. (1994) as the PDF and CDF is given in table (1). In the literature, some extensions of Pareto distribution are available such as the Alpha-Power Pareto (APP) distribution and Generalized Pareto (GP) distribution then the probability density function (PDF) with cumulative distribution function (CDF) are defined in table (1).

Table 1: The PDF and CDF of some kinds of Pareto distribution

distribution	Parameters		Range of X	PDF	CDF
	shape	scale			
Pareto	θ	λ	$x > \lambda \;,\quad \theta, \lambda > 0$	$\dfrac{\theta \lambda^{\theta}}{x^{\theta+1}}$	$1 - \dfrac{\lambda^{\theta}}{x^{\theta}}$
APP	θ	α, λ	$0 < x < \dfrac{\lambda}{\theta} \;, \theta, \alpha > 0 ; \alpha \neq 1$	$\dfrac{\theta \log \alpha \;\alpha^{1-x^{-\theta}}}{(\alpha - 1)} \; x^{-\theta-1}$	$\dfrac{\alpha^{1-x^{-\theta}} - 1}{(\alpha - 1)}$
			$x \geq 1 \;, \theta > 0 ; \alpha = 1$	$\dfrac{\theta}{x^{\theta+1}}$	$1 - x^{-\theta}$
GP	θ	α, λ	$0 < x < \dfrac{\lambda}{\theta} \;,\quad \theta, \lambda > 0$	$\dfrac{1}{\lambda}\left(1 - \dfrac{\theta x}{\lambda}\right)^{\frac{1}{\theta}-1}$	$1 - \left(1 - \dfrac{\theta x}{\lambda}\right)^{\frac{1}{\theta}}$

Figures (1) to (3), demonstrate the graphs of the pdf of Pareto (β, α), APP $(\theta, \lambda, \alpha)$, and GP $(\theta, \lambda, \alpha)$ distributions. Obviously, at different values of the distribution parameters, it's noted that: The pdf of Pareto distribution is decreasing function for $(\beta, \alpha \geq 1)$, and for $(\beta, \alpha \leq 1)$, see Figure (1). The pdf of APP distribution is uni-modal and positively skewed for $(\theta < 0.6 \,, \alpha > 1)$, and it's decreasing function for $(\theta > 0 \,, \alpha < 1)$, see Figure (2). The pdf of GP distribution is increasing for $(\theta, \lambda, \alpha > 1)$, decreasing for $(\theta < 0.1; \;\lambda, \alpha < 1)$, and it's U-shaped for $(\theta > 0.1; \;\lambda, \alpha < 1)$, see Figure (3).

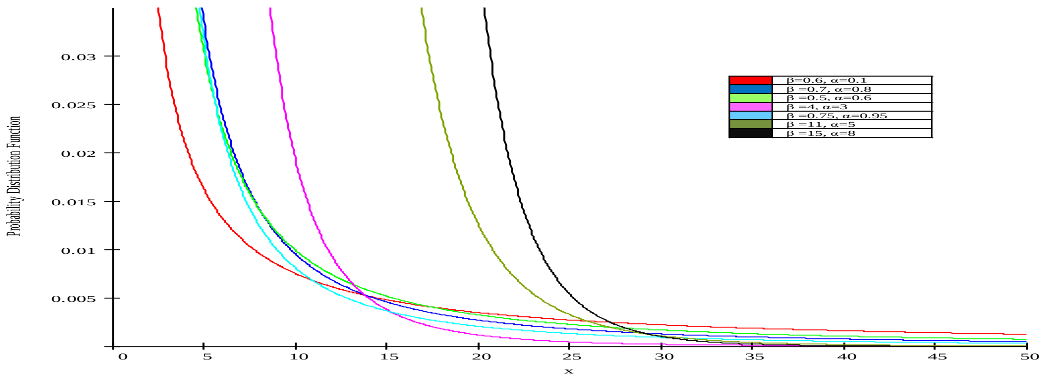


Figure 1: pdf of Pareto Distribution for different values of the parameters

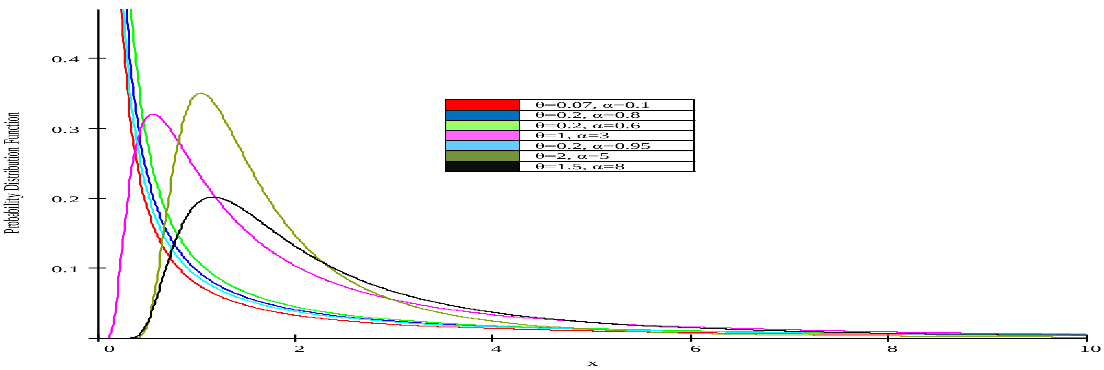


Figure 2: pdf of APP Distribution for different values of the parameters

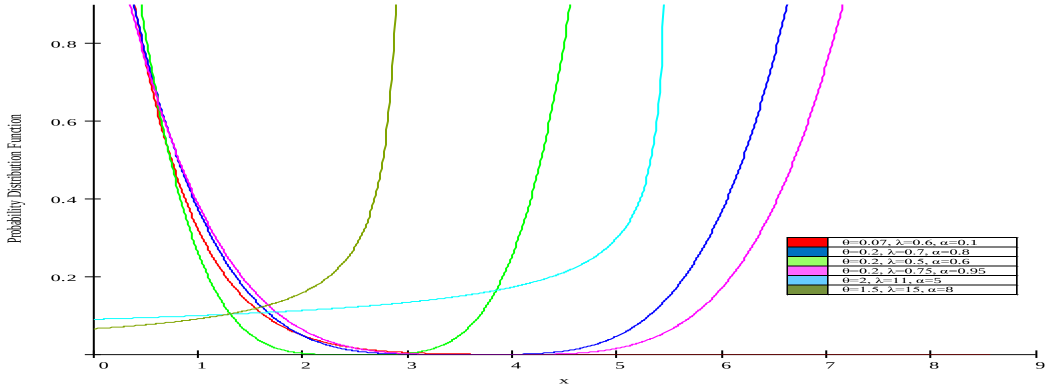


Figure 3: pdf of GP Distribution for different values of the parameters

3. APTGP DISTRIBUTION

The new method called alpha-power transformation (APT) was proposed by Mahdavi and Kundu (2017). They defined the pdf and the cdf of the APT method as follows

$$f(x)=\begin{cases}\left(\dfrac{\log\alpha}{\alpha-1}\right)f^{*}(x)\,\alpha^{F^{*}(x)}; & \alpha>0;\alpha\neq1\\[2ex] f^{*}(x) & ;\alpha=1\end{cases}$$

Where, θ the shape parameter and λ,α the scale parameters. The GP distribution which was defined in table (1), was transformed into a new one using APT method and it's named α-power transformation generalized Pareto distribution and symbolized by (APTGP). In light of APT method, the pdf and cdf of the APTGP distribution with the shape parameter θ and the scale parameter λ are defined as follows:

$$f(x)=\begin{cases}\dfrac{\log\alpha\;\alpha^{1-\left(1-\frac{\theta x}{\lambda}\right)^{\frac{1}{\theta}}}}{\lambda(\alpha-1)}\left(1-\dfrac{\theta x}{\lambda}\right)^{\frac{1}{\theta}-1} & 0<x<\dfrac{\lambda}{\theta}\;,\theta,\lambda,\alpha>0;\alpha\neq1\\[3ex] \dfrac{1}{\lambda}\left(1-\dfrac{\theta x}{\lambda}\right)^{\frac{1}{\theta}-1} & 0<x<\dfrac{\lambda}{\theta}\;,\theta,\lambda>0;\alpha=1\end{cases}$$

$$F(x)=\begin{cases}\dfrac{\alpha^{1-\left(1-\frac{\theta x}{\lambda}\right)^{\frac{1}{\theta}}}-1}{(\alpha-1)} & 0<x<\dfrac{\lambda}{\theta}\;,\theta,\lambda,\alpha>0;\alpha\neq1\\[3ex] 1-\left(1-\dfrac{\theta x}{\lambda}\right)^{\frac{1}{\theta}} & 0<x<\dfrac{\lambda}{\theta}\;,\theta,\lambda>0;\;\alpha=1\end{cases}$$

Figures (4), demonstrate the graph of the pdf of APTGP (θ,λ,α)distribution. Obviously, at different values of the distribution parameters, it's noted that: The pdf of APTGP distribution is increasing for $(\theta,\lambda\geq1;\;\alpha>1)$, decreasing for $(\theta<1,\theta\neq0.2;\;\;\lambda,\alpha<1)$, and it's U-shaped and then decreases be uni-modal positively skewed for $(\theta=0.2;\;\lambda,\alpha<1)$. In addition, Figure (5) illustrate cumulative distribution function (CDF) of APTGP distribution under different values of the parameters (θ,λ,α), the plots of the CDF indicates increasing function, concave in shape extending to the right for $(\theta, \lambda, \alpha > 1)$, and convex in shape extending to the right for $(\theta, \lambda, \alpha < 1)$.

Therefore, In conclusion, it's clear that the proposed distribution (APTGP) is more flexible then the other distributions (Pareto distribution, Alpha-Power Pareto (APP) distribution, Generalized Pareto (GP) distribution).

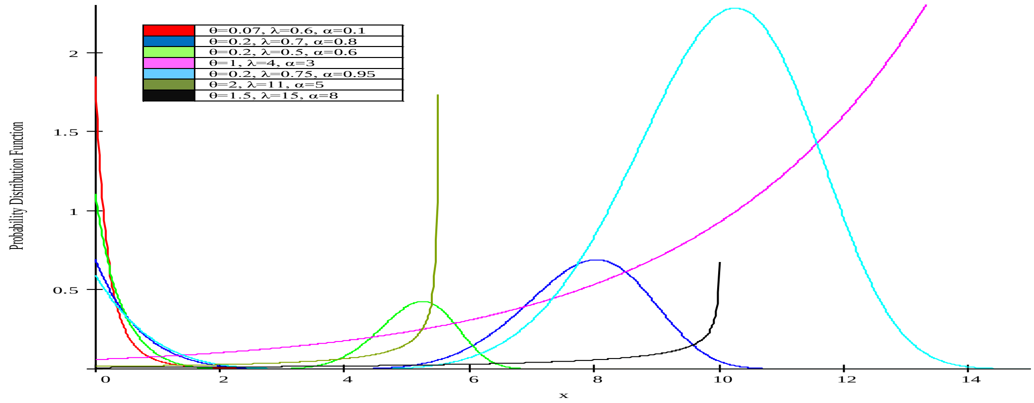


Figure 4: pdf of APTGP Distribution for different values of the parameters

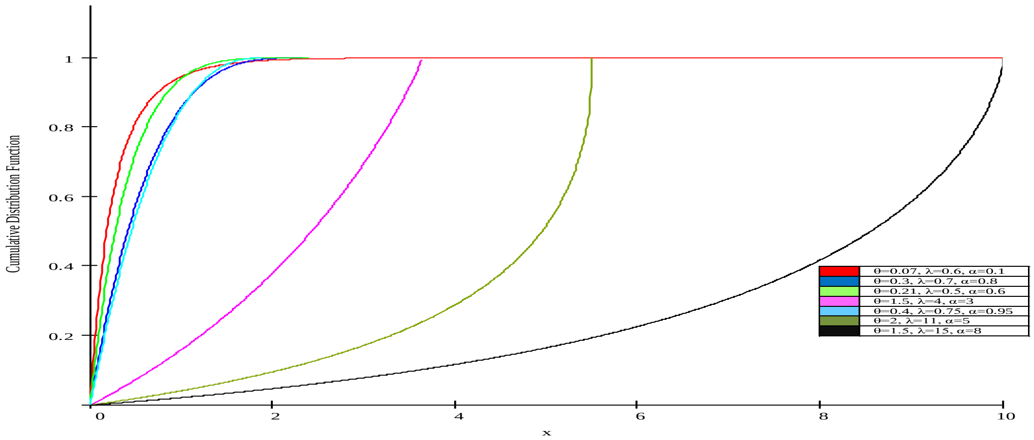


Figure 5: CDF of APTGP Distribution for different values of the parameters

4. SOME STATISTICAL PROPERTIES OF SOME KINDS OF PARETO DISTRIBUTION

Some statistical properties of APTGP, Pareto, APP and GP distribution represented in median, mode, variance, moments, moment generating function, characteristic function, reliability, hazard function are presented in table (2).

Table 2: Some Statistical Properties of some kinds of Pareto distribution

distribution	Pareto	APTGP	APP	GP
Moments	$\frac{\theta\,\lambda^r}{\theta - r}$	$\mu_r' = B\sum_{j=0}^{\infty} C_j\,\gamma(\theta\,j+1, \log\alpha)$ $B = \frac{\alpha\,\lambda^r}{\theta^r(\alpha-1)}$;$C_j = \frac{(-1)^j\binom{r}{j}}{(\log\alpha)^{\theta j}}$; r = 1,2,...	$D0\sum_{K=0}^{\infty} K0_K \sum_{j=0}^{\infty} d0_j$ $D0 = \frac{\alpha\theta}{(1-\alpha)}$; $d0_j = \frac{r!}{(K\theta - r + \theta)}$; $K0_K = \frac{(-\log\alpha)^{K+1}}{K!}$	$\left(\frac{\lambda}{\theta}\right)^r \sum_{j=0}^{\infty}\Phi0_j$ $\Phi0_j = \frac{(-1)^j\binom{\frac{1}{\theta}-1}{j}}{r+j+1}$
arithmetic mean	$\frac{\theta\,\lambda}{\theta - 1}$	$\frac{\alpha\,\lambda}{\theta(\alpha-1)}\sum_{j=0}^{\infty}\Phi_j$; $\Phi_j = \frac{(-1)^j\binom{1}{j}\Psi_j}{(\log\alpha)^{\theta j}}$	$D0\sum_{K=0}^{\infty} K0_K \sum_{j=0}^{\infty} d0_j$, r = 1 $D0 = \frac{\alpha\theta}{(1-\alpha)}$; $d0_j = \frac{r!}{(K\theta - r + \theta)}$; $K0_K = \frac{(-\log\alpha)^{K+1}}{K!}$	$\frac{\lambda}{\theta}\sum_{j=0}^{\infty}\Phi0_j$, r = 1 $\Phi0_j = \frac{(-1)^j\binom{\frac{1}{\theta}-1}{j}}{r+j+1}$

Moment generating function	$\theta\left(-\lambda t\right)^{\alpha}\Gamma\left(-\theta,-\lambda t\right)$	$D\sum_{j=0}^{\infty} d_j \ \gamma\left(\theta j+1,\log\alpha\right)$ $D=\frac{\alpha e^{\frac{\lambda t}{\theta}}}{(\alpha-1)};\ d_j=\frac{(-1)^j(\lambda t)^j}{j!\,\theta^j(\log\alpha)^{\theta j}}$	$D0\sum_{K=0}^{\infty} K0_K \ \sum_{j=0}^{\infty} d0_j$ $D0=\frac{\alpha\theta}{(1-\alpha)};d0_j=\frac{r!}{(K\theta-r+\theta)};K0_K=\frac{(-\log\alpha)^{K+1}}{K!}$	$\frac{1}{\theta}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Phi1_j\Phi1_{kj}$ $\Phi1_j=\frac{(-1)^j\binom{\frac{1}{\theta}-1}{j}}{k+j+1};\Phi1_k=\frac{(t\lambda/\theta)^k}{k!}$
Quantile function	$\lambda(1-p)^{-\theta^{-1}}$	$\frac{\lambda}{\theta}\left(1-(1-A)^{\theta}\right)$, $A=\left(\frac{\log(p(\alpha-1)+1)}{\log\alpha}-1\right)$	$\left(\frac{\log\left(\alpha/\left(p(\alpha-1)+1\right)\right)}{\log\alpha}\right)^{-\theta^{-1}},\alpha\neq1$	$\frac{\lambda}{\theta}\left[1-(1-p)^{\theta}\right]$
RF	$\frac{\lambda^{\theta}}{x^{\theta}}$	$\frac{\alpha}{(\alpha-1)}\left(1-\alpha^{-\left(1-\frac{\theta x}{\lambda}\right)^{\frac{1}{\theta}}}\right)$	$\frac{\alpha\left(1-\alpha^{-x^{-\theta}}\right)}{(\alpha-1)},\alpha\neq1$	$\left(1-\frac{\theta x}{\lambda}\right)^{\frac{1}{\theta}}$
HF	$\frac{\theta}{x}$	$\alpha^{-y^{\frac{1}{\theta}}}y^{\frac{1}{\theta}-1}\left(1-\alpha^{-y^{\frac{1}{\theta}}}\right)^{-1}\lambda^{-1}\log\alpha$ $y=1-\frac{\theta x}{\lambda}$	$\frac{\theta\log\alpha\ \alpha^{-x^{-\theta}}}{1-\alpha^{-x^{-\theta}}}\ x^{-\theta-1},\alpha\neq1$	$\frac{1}{\lambda}\left(1-\frac{\theta x}{\lambda}\right)^{-1}$

4. COMPARISON APTGP DISTRIBUTION WITH OTHER FORMS OF PARETO USING REAL DATA SET

The following data, it's the results of a clinical trial report presented by Freireich *et. al.*, (1963), the data set is: (1,1,2,2,3,4,4,5,5,8,8,88,11,11,12,12,15,17,22,23) is the remission times of 21 acute leukemia patients which took drug 6-MP. Table (3) shows the results of ML estimators for the parameters of the distributions, and the results of the goodness-of-fit tests. Based on the results of table (3), we note that MLEs of the APTGP is the best as it gives the smallest MSEs compared to the rest of the distributions.

Also, the results showed that the data fit to APTGP distribution, as well as to the rest of the distributions, except for generalized Pareto distribution. This indicates that the APTGP distribution is the best in particular compared to the aforementioned competitive models, as the APTGP distribution gave the smallest value for (AIC) and the (MAIC) compared to the rest of the other distributions mentioned.

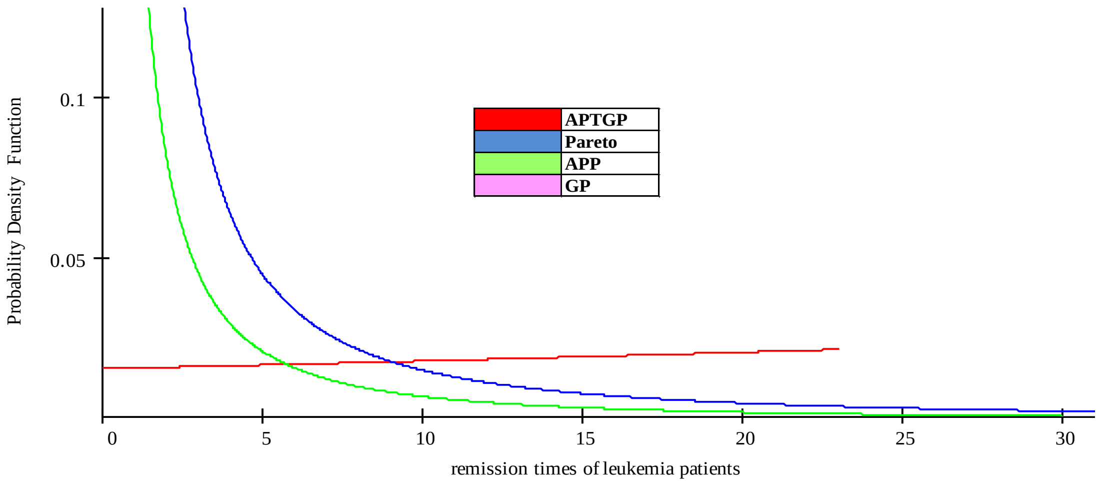
Table 3: results of ML estimators and of the goodness-of-fit tests (the remission times of leukemia patients)

Distributions	goodness of fit criteria			Estimation	parameters				Decision
	K_S	*AIC*	*MAIC*		α	θ	λ	ω	
APTGP	0.0476	139.511	140.9226	mle	1.37357	1.000038	23.001785	0.043477	fit
				mse	0.000012	3.78418E-9	-	-	
Pareto	0.2940	147.92	148.5909	mle	-	0.547909	1	-	fit
				mse	-	-	-	-	
APP	0.2810	145.805	146.4718	mle	1.860061	0.651599	-	-	fit
				mse	0.00361	0.187142	-	-	
GP	0.7814	945.914	946.1241	mle	-	0.043477	-	-	Don't
				mse	-	-	-	-	

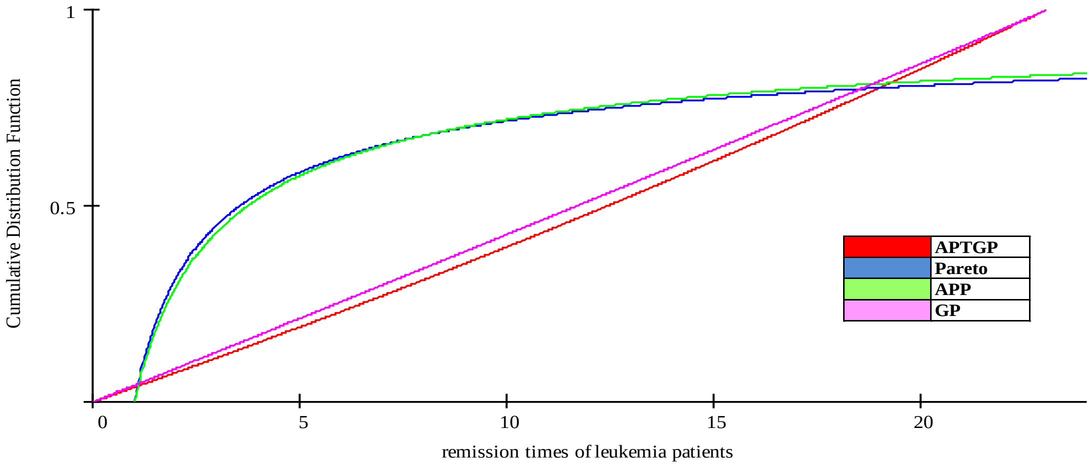
Critical value at 1% = 0.355695

Figures (6) and Figure (7), demonstrate the PDF with CDF graphs of Pareto distribution (β,α), APP distribution (θ,λ,α), GP distribution (θ,λ,α), and APTGP distribution (θ,λ,α). Obviously, it's noted the PDF of Pareto, APP, GP distributions are decreasing functions and it's increasing function APTGP distribution. Also, the plots of the CDF indicates increasing function, concave in shape extending to the right for APP and APTGP distributions. It's convex in shape extending to the right for Pareto, GP distributions. Figure (8) and Figure (9), illustrate Reliability Function (RF) with the Hazard Function (HF). The plots of the RF indicates decreasing function, concave in shape for Pareto, and APP distributions. It's convex in shape for APTGP and GP distributions.

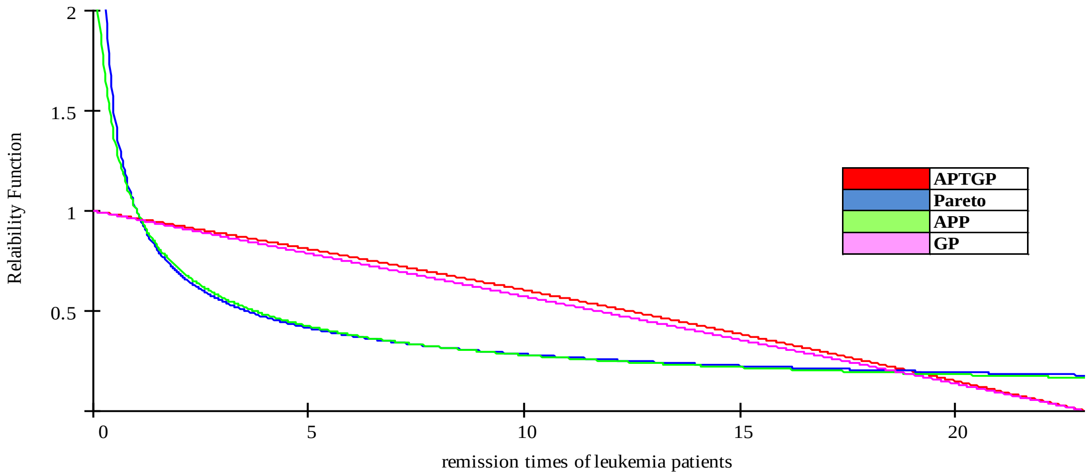
Also, it's noted the HF of Pareto, APP, GP distributions are decreasing functions and it's increasing function of the APTGP distribution.



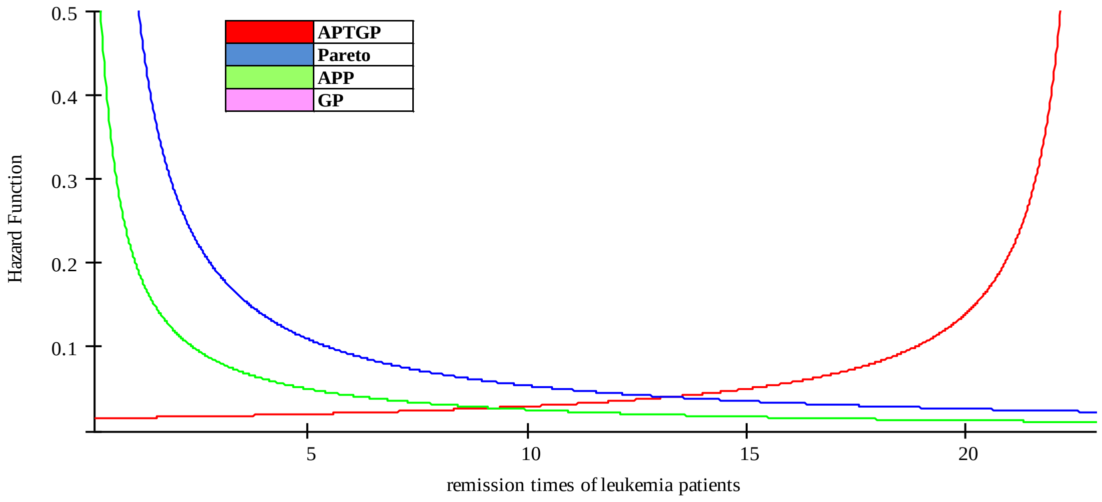
Figures 6: The Shapes of Plots of the Probability Density Function



Figures 7: The Shapes of Plots of the Cumulative Distribution Function



Figures 8: The Shapes of Plots of the Reliability Function



Figures 9: The Shapes of Plots of the Hazard Function

Discussion

In this article, a mathematical model named as named α-power transformation generalized Pareto distribution and symbolized by (APTGP) based on alpha-power transformation (APT) which was proposed by Mahdavi and Kundu (2017) was introduced. Various properties of APTGP distribution are obtained such as moment, quantile function, hazard function, reliability function are studied. Furthermore, a real remission times application of the APTGP distribution to real remission times of acute leukemia patients data sets was presented to illustrate that this distribution provides a better fit than Pareto, APP and GP distributions. The comparison has been shown that the proposed distribution is more flexible than the others.

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