NORMALIZING EXTENSIONS OF RIGHT G-SEMILOCAL AND RIGHT N-SEMILOCAL RINGS

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ABSTRACT

The aim of this paper is to focus on two new classes of rings called right G-semilocal and right N-semilocal. These classes are a generalization of the class of semilocal rings. We also study the transfer of right G-semilocality and right N-semilocality from a ring R to its normalizing extension S and vice versa.

1. Introduction

Throughout this paper, let S be a finite normalizing extension of a ring R, that is S is finitely generated as an R-module by elements $x_1, x_2, \dots, x_n$ of S with $Rx_i = x_iR$ for $i = 1,2, \dots, n$, $J(R)$ and $J(S)$ denote the Jacobson radicals of R and S respectively.

Recall that a ring R is said to be semilocal if $\overline{R} = R/J(R)$ is an Artinian ring, R is said to be semiprimary if $R/J(R)$ is an Artinian and $J(R)$ is nilpotent ideal, R is said to be right perfect if $R/J(R)$ is an Artinian and $J(R)$ is right T-nipotent, a ring R is called right Noetherian if it satisfies the ascending chain condition on right ideals and a ring R is called right Goldie if R has finite right uniform dimension and R satisfies the ascending chain condition on right annihilators.

Several authors studied the transfer of algebraic properties from S to R and from R to S. Resco (1981) proved that if S is right Artinian, semiprimary or perfect, then so is R. Moreover, Lanski (1980) proved that when S is a right Goldie ring, R is also a right Goldie ring. Conversely, if R is a semiprime right Goldie ring and S is a prime ring, then S is a right Goldie ring.

The first obstacle we meet on our way to extend the property of Artinianity of the quotient ring $R/J(R)$ to the Noetherian property, or more generally to the Goldie one, is that-unlike the Artinian property the latter two are not left-right

symmetric. There are examples of semiprimitive right Noetherian (resp., right Goldie) rings which are not left Noetherian (resp., left Goldie).
Throughout this paper, we study the properties of the right N-semilocal (resp., G-semilocal) and the left N-semilocal (resp., G-semilocal) is analogously. By an N-semilocal (resp., G-semilocal) ring we mean a ring that is both left and right N-semilocal (resp., G-semilocal) ring.

2. Right N-semilocal rings

Definition 2.1. A ring R is called right N-semilocal if $R/J(R)$ is a right Noetherian ring. Since the homomorphic image of a right Notherian ring is a right Noetherian, and every right Artinian is right Noetherian therefore, we conclude that every right Noetherian or semilocal ring is right N-semilocal. But the following example illustrates that the converse is not true.

Example 2.2. Consider the ring R of upper triangular 2×2 metrices $\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{pmatrix} = R$. Then R is neither left nor right Noetherian, since R has the infinite ascending chains $I_1 \subset I_2 \subset \cdots \subset I_i \subset \cdots$ and $J_1 \subset J_2 \subset \cdots \subset J_i \subset \cdots$ of left and right ideals in R respectively, where $I_i = \left\{\begin{pmatrix} \mathbb{Z} & T_i \\ 0 & 0 \end{pmatrix}\right\}$, $J_i = \left\{\begin{pmatrix} 0 & T_i \\ 0 & \mathbb{Z} \end{pmatrix}\right\}$ and $T_i = Q_{p^i} = \left\{\frac{n}{p^i} : n \in \mathbb{Z}, \text{p is a fixed prime number}\right\}$. However, $J(R) = \left\{\begin{pmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{pmatrix}\right\}$ is a nilpotent ideal, where $J^2(R) = 0$ and $R/J(R) \cong \mathbb{Z} \oplus \mathbb{Z}$. Then $R/J(R)$ is a commutative Noetherian ring, but not Artinian, hence R is N-semilocal which is not semilocal. Moreover, R is neither left nor right Noetherian.

We need the following two results.

Theorem 2.3. (Mcconnell and Robson,1987, Corollary 10.1.11). S is right Noetherian if and only if R is right Noetherian.

Theorem 2.4. (Fahmy et al., 2012, Lemma 2.1). Let S be a finite normalizing extension of R. Then $S/J(S)$ is a finite normalizing extension of $R/J(R)$ whose normal generators are $\bar{x}_i = x_i + J(S)$.

Now we prove that the right N-semilocality transfers from R to S and vice versa.

Theorem 2.5. Let S be a finite normalizing extension of R. Then S is a right N-semilocal ring if and only if R is a right N-semilocal ring.

Proof. S is a right N-semilocal ring means that $\bar{S} = S/J(S)$ is a right Noetherian ring. By Theorem 2.4 $\bar{S}$ is a finite normalizing extension of $\bar{R} = R/J(R)$. Therefore, by Theorem 2.3, $\bar{S}$ is a right Noetherian ring if and only if $\bar{R}$ is.

Definition 2.6. A ring R is called right N-right perfect if $R/J(R)$ is right Noetherian and $J(R)$ is right T-nilpotent ideal.

In Example 2.2, we have seen that the Jacobson radical is a nilpotent ideal, hence $J(R)$ is right and left T-nilpotent. Therefore Example 2.2 services as an N-semiprimary ring and as an N-right (or left) perfect ring.

Proposition 2.7. If S is right N-right perfect, then so is R.

Proof. Since $J(R) = J(s) \cap R$ the right T-nilpotency of $J(S)$ implies that $J(R)$ is right T-nilpotent, and the fact that $S/J(S)$ is right Noetherian implies that $R/J(R)$ is right Noetherian, hence R is right N-right perfect.

Definition 2.8. A ring R is called right N-semiprimary if $R/J(R)$ is right Noetherian and $J(R)$ is nilpotent ideal.

The proof of the following proposition is similar to the proof of Proposition 2.7.

Proposition 2.9. If S is right N-semiprimary, then so is R.

3. Right G-semilocal rings.

Definition 3.1. A ring R is called right G-semilocal if $R/J(R)$ is a right Goldie ring.

Since every right Noetherian ring is right Goldie, so it is clear from the definitions 2.1 and 3.1 that every right N-semilocal is right G-semilocal. The following example shows that the class of right N-semilocal rings is a proper subclass of the class of right G-semilocal rings.

Example 3.2. Let A and B be rings such that ${}_A B_A$ is an A-bialgebra and let $R = T(A, B)$ be the trivial extension of A over B. $T(A, B)$ can be represented as the set of upper triangular matrices $\left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a \in A, b \in B\right\}$, then $J(R) = \left\{\begin{pmatrix} j & b \\ 0 & j \end{pmatrix} ; j \in J(A), b \in B\right\}$ and $\overline{R} = R/J(R) \cong \left\{\begin{pmatrix} \bar{a} & 0 \\ 0 & \bar{a} \end{pmatrix} ; \bar{a} = a + J(R)\right\} \cong \overline{A} = A/J(A)$. If we take $A = \mathbb{Z}[x_1, x_2, \dots]$, $B = \mathbb{Q}[x_1, x_2, \dots]$, then $R/J(R) \cong \mathbb{Z}[x_1, x_2, \dots]$ which is Goldie but not Noetherian. Thus R is G-semilocal which is not N-semilocal.

The following result is crucial to the sequel.

Theorem 3.3. (Lanski, 1980, Theorem 2). Let S be a finite normalizing extension of a ring R. If S is right Goldie ring, then R is a right Goldie ring.

Thus, we conclude the next theorem.

Theorem 3.4. Let S be a finite normalizing extension of a ring R. If S is a right G-semilocal ring, then so is R.

Proof. Since $\overline{S} = S/J(S)$ is a right Goldie, and $\overline{S}$ is a finite normalizing extension of $\overline{R} = R/J(R)$ by Theorem 2.4, thus $\overline{R}$ is a right Goldie ring by Theorem 3.3. Hence R is a right G-semilocal ring.

Definition 3.5. A ring R is called right G-right perfect if $R/J(R)$ is right Goldie and $J(R)$ is right T-nilpotent ideal.

Proposition 3.6. If S is right G-right perfect, then so is R.

Proof. Since $J(R) = J(s) \cap R$ the right T-nilpotency of $J(S)$ implies that $J(R)$ is right T-nilpotent, and $S/J(S)$ is right Goldie implies that $R/J(R)$ is right Goldie by Theorem 3.4. Thus R is right G-right perfect.

Proposition 3.7. If S is right G-semiprimary, then so is R.

Proof. Using the same procedure as in the proof of proposiyion 3.6 and the relation $J(R) = J(s) \cap R$ the result follows.

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