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Concomitants of k-record values based on Sarmanov exponential bivariate distribution

G. M. Mansour, H. M. Barakat, M. A. Alawady, and I. A. Husseiny
Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt

Corresponding author: E-mail: ghadamansour776@gmail.com

***ABSTRACT* : When prior information is in the form of marginal distributions, it is advantageous to consider families of bivariate distributions with specified marginals when modelling bivariate data. Among these families of bivariate distributions, the Farlie-Gumbel-Morgenstern (FGM) family was studied extensively by many authors. Several modifications to the FGM family have been proposed in the literature to increase the range of the correlation between its marginals. One important limitation of the FGM family is that the correlation coefficient between its marginals is restricted to a narrow range $\left[-\frac{1}{3},\frac{1}{3}\right]$. One of the most pliable and robust extensions of the classical FGM family of bivariate distributions is the Sarmanov family, which was proposed and used by Sarmanov (1974) as a new model of hydrological processes, inter alia. Despite the salient and almost unique features of this family, it is never used in the literature. We study the concomitants of k −record values based on Sarmanov exponential distribution. Also, we derived the joint distribution of concomitants of k −record values based on this family.**

KEYWORDS: Sarmanov family; FGM family; Concomitants; K-Record values; Exponential distribution.

Mathematical Subject Classification: 60B12, 62G30.

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I. INTRODUCTION

The concept of record values was introduced by Chandler (1952). Since then, the literature has devoted a great deal of attention to record values and their applicability in a variety of contexts (cf. Ahsanullah, 2009, Bdair and Raqab, 2013, and Nevzorov and Ahsanullah, 2000). Sports, seismology, and athletic activities, as well as industrial stress testing, meteorological analysis, and oil, hydrological, and mining surveys, can all benefit from record values.

Let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed random variables (RVs) with a common continuous distribution function (DF) $F_X(x)$ and probability density function (PDF) $f_X(x)$. An observation X_j is called an upper record value if its value exceeds that of all previous observations. Thus, X_j is an upper record if $X_j > X_i$ for every $i < j$. An analogous definition can be given for lower record values. The model of record values becomes inadequate in several situations when the expected waiting time between two record values is very large.

For example, serious difficulties for the statistical inference based on records arise because record data is extremely rare in practical contexts and the predicted waiting time for each record following the first is infinite. In those situations, second or third largest values are of great importance. These issues are avoided, however, if we consider Dziubdziela and Kopocinski's (1976) k −record value model, refer also to Amany et al. (2019), Berred (1995), and Fashandi and Ahmadi (2012).

The definition of nth upper k-record is given by Dziubdziela and Kopocinski (1976), as follows:

For a positive integer k, define $T_{1,k} = k$ and for $n \geq 2$, $T_{n,k} = \min\{j: j > T_{n-1,k}, X_{j-k+1:j} > X_{T_{n-1,k}-k+1:T_{n-1,k}}\}$. Let $Z_{n,k} = X_{T_{n,k}-k+1:T_{n,k}}, n \geq 1$, where $X_{i:n}$ denotes the ith order statistic in a sample of size n. The sequence $\{X_{n,k}, n \geq 1\}$ is referred to as the sequence of upper k −record values. Therefore, this sequence is the sequence of the kth largest observations yet seen. In an analogous way, a lower k −record value can be defined. If we put $k = 1$, the results for usual records can be obtained as a special case.

The PDF of the nth upper and lower k-record values are respectively given by

$$g_{n,k}^{(u)}(x) = \frac{k^n}{\Gamma(n)}(-\log(\overline{F}_X(x)))^{n-1}(\overline{F}_X(x))^{k-1} f_X(x)$$

and

$$g_{n,k}^{(\ell)}(x) = \frac{k^n}{\Gamma(n)}(-\log(F_X(x)))^{n-1}(F_X(x))^{k-1} f_X(x),$$

where $\Gamma(.)$ is the usual gamma function and $\overline{F}_X(x) = 1 - F_X(x)$. Moreover, the joint PDF of $X_{n,k}^{(u)}$ and $X_{m,k}^{(u)}$ is given by

$$g_{m,n,k}^{(u)}(x_1, x_2) = \frac{k^n}{\Gamma(n)\Gamma(m-n)}(-\log(\overline{F}_X(x_1)))^{n-1}\left(\overline{F}_X(x_2)\right)^{k-1}$$

$$\times \left(-\log\frac{\overline{F}_X(x_2)}{\overline{F}_X(x_1)}\right)^{m-n-1} \frac{f_X(x_1)f_X(x_2)}{\overline{F}_X(x_1)}, x_1 \leq x_2.$$

When prior information is in the form of marginal distributions, it is advantageous to consider families of bivariate distributions with specified marginals when modelling bivariate data. Among these families of bivariate distributions, the Farlie-Gumbel-Morgenstern (FGM) family was studied extensively by many authors. This family is represented by the bivariate DF $F_{X,Y}(x,y) = F_X(x)F_Y(y)[1 + \theta(1 - F_X(x))(1 - F_Y(y))], -1 \leq \theta \leq 1$, where $F_X(x)$ and $F_Y(y)$ are the marginal DF's of the two RVs X and Y, respectively. One important limitation of the FGM family is that the correlation coefficient between its marginals is restricted to a narrow range $\left[-\frac{1}{3}, \frac{1}{3}\right]$. Accordingly, this family can be used to model the bivariate data that exhibits low correlation. For some applications of the FGM family, we refer to Drouet and Kotz (2001) and Schucany et al. (1978) and the references therein.

Several modifications to the FGM family have been proposed in the literature to increase the range of the correlation between its marginals. Among them, Cambanis (1977) proposed a three-parameter distribution family as a natural generalization of the multivariate FGM family. This family has the advantage of containing all conditional distributions of the multivariate FGM family, but it marginally outperforms the FGM in terms of the correlation coefficient. Furthermore, unlike the FGM family, the Cambanis family's marginals are not F_X and F_Y, but are determined uniquely by them. For more details about this family, we refer to Abd Elgawad et al. (2021b), Alawady et al. (2021b), and Nair et al. (2016). After few years, Huang and Kotz (1984) used successive iterations in the FGM family to increase the correlation between components. They demonstrated that for certain marginals, just one iteration can triple the covariance. The maximum positive value of the correlation coefficient in the two-parameter single-iterated FGM family when the marginals are uniform is 0.434. Recently, this family was studied in different aspects by Alawady et al. (2020), Barakat and Husseiny (2021), and Barakat et al. (2021).

Once again, two extended families were proposed by Huang and Kotz (1999), in which an extra shape parameter is employed as an exponent of marginal DFs or their survival functions. These families have the highest positive correlations of 0.375 and 0.391. For more details about these extensions, we refer to Abd Elgawad et al. (2020, 2021c), Bairamov and Kotz (2002), and Fischer and Klein (2007). Bairamov et al. (2001) proposed a five-parameter family as the most general form of the FGM family. For uniform marginals, the correlation coefficient of this extension is in the range $(-0.48, 0.502)$. For some recent works about this family and its properties, see Alawady et al. (2021a). Bekrizadeh et al. (2012) presented an extension for the FGM family by adding two new shape factors and demonstrated that the strongest positive Spearman's correlation coefficient between the marginals is 0.43. For some recent works pertain this family, see Abd Elgawad et al. (2021a).

Sarmanov (1974) proposed a new mathematical model of hydrological processes described by the following family of bivariate DFs

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)[1 + 3\alpha\bar{F}_X(x)\bar{F}_Y(y)$$

$$+5\alpha^2(2F_X(x)-1)(2F_Y(y)-1)\bar{F}_X(x)\bar{F}_Y(y)], |\alpha| \leq \frac{\sqrt{7}}{5}, \quad (1.1)$$

denoted by SAR(α). The corresponding PDF is given by

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)[1 + 3\alpha(2F_X(x)-1)(2F_Y(y)-1)$$

$$+\frac{5}{4}\alpha^2(3(2F_X(x)-1)^2-1)(3(2F_Y(y)-1)^2-1)]. \quad (1.2)$$

The Sarmanov copula's correlation coefficient is α. As a result, this copula's maximum correlation coefficient is 0.529 (cf. Balakrishnan and Lin, 2009; page 74).

All of the foregoing extended families, including the Sarmanov family stated in (1.1) and (1.2), are special examples of an extended family to the FGM family, which is defined via the PDF

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)[1 + \Psi(x,y;\overline{\theta})], \quad (1.3)$$

where f_X and f_Y are the PDFs of F_X and F_Y, respectively, $\overline{\theta}$ is a shape-parameter vector, $1 + \Psi(x,y;\overline{\theta}) \geq 0$, and Ψ is a measurable function satisfying $E\,(\Psi(X,y;\overline{\theta})) = E\,(\Psi(x,Y;\overline{\theta})) = 0$. This feature leads us to regard the Sarmanov family as an extension of the FGM family, despite the shape parameter α has no value that makes the family turns to the FGM family. On the other hand, the PDF (1.3) is a slight modification of the Sarmanov PDF, which was initiated by Sarmanov (1966), where $\Psi(x,y;\overline{\theta}) = \theta\psi_1(x)\psi_2(y)$, θ is a shape parameter, $\psi_1(.)$ and $\psi_2(.)$ are measurable functions fulfilling $1 + \theta\psi_1(x)\psi_2(y) \geq 0$, and $E\,(\psi_1(X)) = E\,(\psi_2(Y)) = 0$.

Among the above-mentioned families, the Sarmanov family is the most efficient. On both the positive and negative sides, it delivers the best improvement in the correlation level. Furthermore, among all known extensions, this family contains only one shape parameter, which is shared by the two marginal variates. The last feature makes it easier to estimate the shape parameter α, in the Sarmanov copula, by using the sample correlation estimate. Despite all of these distinguishing characteristics, this family has never been used since its inception. In the present paper, we reveal some additional motivating properties for the Sarmanov family. We also discuss some aspects of the concomitants of k −record values. Recently, Barakat et al. (2022) and Husseiny et al. (2022) studied some aspects of concomitants of OSs and record values from this family.

Let $(X_i, Y_i), i = 1,2,\ldots$, be a random bivariate sample, with a common continuous DF $F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$. When the investigator is just interested in studying the sequence of k −records of the first component X, the second component associated with the k −record value of the first one is termed as the concomitant of that k −record value. The subject of k −record values and their concomitants can be found in a wide range of practical experiments, e.g., see Amany et al. (2019), Bdair and Raqab (2013), Chacko and Mary (2013), Chacko and Muraleedharan (2018), and Thomas et al. (2014). The PDFs of the concomitants $Y_{[n,k]}^{(u)}$ (the nth upper concomitant of $X_{n,k}^{(u)}$) and $Y_{[n,k]}^{(\ell)}$ (the nth lower concomitant of $X_{n,k}^{(\ell)}$) are given by

$$f_{[n,k]}^{(u)}(y) = \int_0^\infty f_{Y|X}(y|x) g_{n,k}^{(u)}(x) dx \quad and \quad f_{[n,k]}^{(\ell)}(y) = \int_0^\infty f_{Y|X}(y|x) g_{n,k}^{(\ell)}(x) dx, \tag{1.4}$$

respectively, where $f_{Y|X}(y|x)$ is the conditional PDF of Y given X. Moreover, the joint PDF of concomitants $Y_{[n,k]}^{(u)}$ and $Y_{[m,k]}^{(u)}$ $(n < m)$ is given by

$$f_{[n,m,k]}^{(u)}(y_1, y_2) = \int_0^\infty \int_{x_1}^\infty f_{Y|X}(y_1|x_1) f_{Y|X}(y_2|x_2) g_{m,n,k}^{(u)}(x_1, x_2) dx_2 dx_1. \tag{1.5}$$

The rest of the paper is organized as follows. We obtain some new interesting results pertaining to the Sarmanov exponential distribution and concomitants of k-record values. Furthermore, the joint distribution of concomitants of k-record values based on the Sarmanov exponential distribution is studied.

II. Marginal distributions of concomitants of k −record values

The PDF of $Y_{[n,k]}^{(u)}$ is represented in the following theorem in a useful way. For a given RV X and some DF F, we will now use the notation $X \sim F$, which signifies that X is distributed as F.

Theorem 1 Let $V_1 \sim F_Y^2$ and $V_2 \sim F_Y^3$. Then

$$f_{[n,k]}^{(u)}(y) = \left(1 - 3\Lambda_{n,k:1}^{(\alpha)} + \frac{5}{2}\Lambda_{n,k:2}^{(\alpha)}\right)\theta_2 \exp(-\theta_2 y) + 2\left(3\Lambda_{n,k:1}^{(\alpha)} - \frac{15}{2}\Lambda_{n,k:2}^{(\alpha)}\right)\theta_2 \exp(-\theta_2 y)(1 - \exp(-\theta_2 y))$$

$$+15\Lambda_{n,k:2}^{(\alpha)}\theta_2 \exp(-\theta_2 y)(1 - \exp(-\theta_2 y)^2), \tag{2.1}$$

where $\Lambda_{n,k:1}^{(\alpha)} = \alpha\left[1 - 2\left(\frac{k}{k+1}\right)^n\right]$ and $\Lambda_{n,k:2}^{(\alpha)} = \alpha^2\left[12\left(\left(\frac{k}{k+2}\right)^n - \left(\frac{k}{k+1}\right)^n\right) + 2\right]$.

Proof. by using (??) we get

$$f_{[n,k]}^{(u)}(y) = \int_0^\infty \theta_2 \exp(-\theta_2 y)\{1 + 3\alpha(2(1 - \exp(-\theta_1 x)) - 1)(2(1 - \exp(-\theta_2 y)) - 1)$$

$$+\frac{5}{4}\alpha^2[3(2(1 - \exp(-\theta_1 x)) - 1)^2 - 1][3(2(1 - \exp(-\theta_2 y)) - 1)^2 - 1]\}$$

$$\times \frac{k^n}{\Gamma(n)}(-\log(\exp(-\theta_1 x)))^{n-1}(\exp(-\theta_1 x))^{k-1}\theta_1 \exp(-\theta_1 x) dx.$$

$$= \int_0^\infty \{\theta_2 \exp(-\theta_2 y) + 3\alpha(2(1 - \exp(-\theta_1 x)) - 1)(2\theta_2 \exp(-\theta_2 y)(1 - \exp(-\theta_2 y)))$$

$-\theta_2\exp(-\theta_2 y)) + \frac{5}{4}\alpha^2[3(2(1-\exp(-\theta_1 x)) - 1)^2 - 1][12\theta_2\exp(-\theta_2 y)(1 - \exp(-\theta_2 y)^2) \quad -12\theta_2\exp(-\theta_2 y)(1-\exp(-\theta_2 y)) + 2\theta_2\exp(-\theta_2 y)]\}\frac{k^n}{\Gamma(n)}(-\mathrm{log}\exp(-\theta_1 x))^{n-1}(\exp(-\theta_1 x))^{k-1} \times \theta_1\exp(-\theta_1 x)dx.$

$$= \theta_2\exp(-\theta_2 y) + 6\theta_2\exp(-\theta_2 y)(1-\exp(-\theta_2 y)) - \theta_2\exp(-\theta_2 y))I_1$$

$$+\frac{5}{4}[12\theta_2\exp(-\theta_2 y)(1-\exp(-\theta_2 y)^2) - 12\theta_2\exp(-\theta_2 y)(1-\exp(-\theta_2 y)) + 2\theta_2\exp(-\theta_2 y)]I_2,$$

where

$$I_1 = \frac{\alpha k^n}{\Gamma(n)}\int_0^\infty (2(1-\exp(-\theta_1 x)) - 1)(-\mathrm{log}\exp(-\theta_1 x))^{n-1}(\exp(-\theta_1 x))^{k-1}\theta_1\exp(-\theta_1 x)dx$$

$$= \alpha\left[1 - 2\left(\frac{k}{k+1}\right)^n\right] = \Lambda_{n,k:1}^{(\alpha)}$$

and

$$I_2 = \frac{\alpha^2 k^n}{\Gamma(n)}\int_0^\infty [3(2(1-\exp(-\theta_1 x)) - 1)^2 - 1](-\mathrm{log}\exp(-\theta_1 x))^{n-1}(\exp(-\theta_1 x))^{k-1}\theta_1\exp(-\theta_1 x)dx$$

$$= \alpha^2\left[12\left(\left(\frac{k}{k+2}\right)^n - \left(\frac{k}{k+1}\right)^n\right) + 2\right] = \Lambda_{n,k:2}^{(\alpha)}.$$

This completes the proof.

Remark 2.1 The case $k = 1$, which mainly includes the case of record values, was handled by Husseiny et al. (2022)

$$f_{[n]}^{(u)}(y) = \left(1 - 3\Lambda_{n:1}^{(\alpha)} + \frac{5}{2}\Lambda_{n:2}^{(\alpha)}\right)\theta_2\exp(-\theta_2 y) + 2\left(3\Lambda_{n:1}^{(\alpha)} - \frac{15}{2}\Lambda_{n:2}^{(\alpha)}\right)\theta_2\exp(-\theta_2 y)(1-\exp(-\theta_2 y))$$

$$+15\Lambda_{n:2}^{(\alpha)}\theta_2\exp(-\theta_2 y)(1-\exp(-\theta_2 y)^2),$$

where $\Lambda_{n:1}^{(\alpha)} = \alpha(1 - 2^{-(n-1)})$ and $\Lambda_{n:2}^{(\alpha)} = \alpha^2(12(3^{-n} - 2^{-n}) + 2)$.

3 Joint distribution of bivariate concomitants of k −record values based on SAR(α)

The following theorem gives the joint PDF $f_{[n,m,k]}^{(u)}(y_1, y_2)$ (defined by (??)) of the concomitants of $Y_{[n,k]}^{(u)}$ and $Y_{[m,k]}^{(u)}, n < m$, in Sarmanov exponential distribution.

Theorem 2 Let $V_1 \sim F_Y^2$ and $V_2 \sim F_Y^3$. Then

$$f_{[n,m,k]}^{(u)}(y_1, y_2) = \theta_2\exp(-\theta_2 y_1)\theta_2\exp(-\theta_2 y_2) + \left(3\Lambda_{n,k:1}^{(\alpha)} - \frac{5}{2}\Lambda_{n,k:2}^{(\alpha)}\right)\theta_2\exp(-\theta_2 y_2)(2\theta_2\exp(-\theta_2 y_1)$$

$$\times (1-\exp(-\theta_2 y_1)) - \theta_2\exp(-\theta_2 y_1)) + \left(3\Lambda_{m,k:1}^{(\alpha)} - \frac{5}{2}\Lambda_{m,k:2}^{(\alpha)}\right)\theta_2\exp(-\theta_2 y_1)$$

$$\times (2\theta_2\exp(-\theta_2 y_2)(1-\exp(-\theta_2 y_2)) - \theta_2\exp(-\theta_2 y_2)) + 5\Lambda_{n,k:2}^{(\alpha)}\theta_2\exp(-\theta_2 y_2)$$

$$\times\left(3\theta_2\exp(-\theta_2y_1)(1-\exp(-\theta_2y_1)^2)-2\theta_2\exp(-\theta_2y_1)(1-\exp(-\theta_2y_1))\right)$$

$$+5\Lambda_{m,k:2}^{(\alpha)}\theta_2\exp(-\theta_2y_1)(3\theta_2\exp(-\theta_2y_2)(1-\exp(-\theta_2y_2)^2)-2\theta_2\exp(-\theta_2y_2)$$

$$\times(1-\exp(-\theta_2y_2)))+\left(9\Lambda_{n,m,k:1}^{(\alpha)}-\frac{15}{2}\Lambda_{n,m,k:2}^{(\alpha)}-\frac{15}{2}\Lambda_{n,m,k:3}^{(\alpha)}+\frac{25}{4}\Lambda_{n,m,k:4}^{(\alpha)}\right)$$

$$\times(2\theta_2\exp(-\theta_2y_1)(1-\exp(-\theta_2y_1))-\theta_2\exp(-\theta_2y_1))(2\theta_2\exp(-\theta_2y_2)(1-\exp(-\theta_2y_2))$$

$$-\theta_2\exp(-\theta_2y_2))+\left(15\Lambda_{n,m,k:2}^{(\alpha)}-\frac{25}{2}\Lambda_{n,m,k:4}^{(\alpha)}\right)(3\theta_2\exp(-\theta_2y_1)(1-\exp(-\theta_2y_1)^2)$$

$$-2\theta_2\exp(-\theta_2y_1)(1-\exp(-\theta_2y_1)))(2\theta_2\exp(-\theta_2y_2)(1-\exp(-\theta_2y_2))-\theta_2\exp(-\theta_2y_2))$$

$$+\left(15\Lambda_{n,m,k:3}^{(\alpha)}-\frac{25}{2}\Lambda_{n,m,k:4}^{(\alpha)}\right)(2\theta_2\exp(-\theta_2y_1)(1-\exp(-\theta_2y_1))\theta_2\exp(-\theta_2y_1))$$

$$\times(3\theta_2\exp(-\theta_2y_2)(1-\exp(-\theta_2y_2)^2)-2\theta_2\exp(-\theta_2y_2)(1-\exp(-\theta_2y_2)))+25\Lambda_{n,m,k:4}^{(\alpha)}$$

$$\times(3\theta_2\exp(-\theta_2y_1)(1-\exp(-\theta_2y_1)^2)-2\theta_2\exp(-\theta_2y_1)(1-\exp(-\theta_2y_1)))(3\theta_2\exp(-\theta_2y_2)$$

$$\times(1-\exp(-\theta_2y_2)^2)-2\theta_2\exp(-\theta_2y_2)(1-\exp(-\theta_2y_2))),$$

where $\Lambda_{m,k:1}^{(\alpha)}$ and $\Lambda_{m,k:2}^{(\alpha)}$ are defined via Theorem 2.2. by replacing n with m in $\Lambda_{n,k:1}^{(\alpha)}$ and $\Lambda_{n,k:2}^{(\alpha)}$, respectively,

$$\Lambda_{n,m,k:1}^{(\alpha)}=\alpha^2\left[-2\left(\frac{k^n}{(k+1)^m}+\frac{k^n}{k^{m-n}(k+1)^n}\right)+\frac{4k^n}{(k+2)^n(k+1)^{m-n}}+1\right],$$

$$\Lambda_{n,m,k:2}^{(\alpha)}=\alpha^3\left[4\left(k^{n-m}-\frac{k^n}{(k+1)^m}\right)-24\left(\frac{k^n}{(k+3)^n(k+1)^{m-n}}-\frac{k^n}{(k+2)^n(k+1)^{m-n}}\right)\right.$$

$$\left.-12\left(\frac{k^n}{(k+1)^nk^{m-n}}-\frac{k^n}{(k+2)^nk^{m-n}}\right)-2\right],$$

$$\Lambda_{n,m,k:3}^{(\alpha)}=\alpha^3\left[4\left(k^{n-m}-\frac{k^n}{(k+1)^nk^{m-n}}\right)-24\left(\frac{k^n}{(k+3)^n(k+2)^{m-n}}-\frac{k^n}{(k+2)^n(k+1)^{m-n}}\right)\right.$$

$$\left.-12\left(\frac{k^n}{(k+1)^m}-\frac{k^n}{(k+2)^m}\right)-2\right],$$

and

$$\Lambda_{n,m,k:4}^{(\alpha)}=\alpha^4\left[4-144\left(\frac{k^n}{(k+3)^n(k+2)^{m-n}}+\frac{k^n}{(k+3)^n(k+1)^{m-n}}-\frac{k^n}{(k+4)^n(k+2)^{m-n}}\right.\right.$$

$$\left.\left.-\frac{k^n}{(k+2)^n(k+1)^{m-n}}\right)-24\left(\frac{k^n}{(k+1)^m}-\frac{k^n}{(k+2)^m}+\frac{k^n}{(k+1)^nk^{m-n}}-\frac{k^n}{(k+2)^nk^{m-n}}\right)\right].$$

Proof. Consider the integration

$$I_{p,q}(n,m,k)=\frac{k^n}{\Gamma(n)\Gamma(m-n)}\int_0^\infty\int_{x_1}^\infty(1-\exp(-\theta_1x_1))^p(1-\exp(-\theta_1x_2))^q(-\log(\exp(-\theta_1x_1))^{n-1}\times(\exp(-\theta_1x_2))^{k-1}\left(-\log\frac{\exp(-\theta_1x_2)}{\exp(-\theta_1x_1)}\right)^{m-n-1}\frac{\theta_1\exp(-\theta_1x_1)\theta_1\exp(-\theta_1x_2)}{\exp(-\theta_1x_1)}dx_2dx_1.$$

Taking the transformation $u_1=-\log(\exp(-\theta_1x_1))$ and $u_2=-\log(\exp(-\theta_1x_2))$, we get

$$I_{p,q}(n,m,k)=\frac{k^n}{\Gamma(n)\Gamma(m-n)}\int_0^\infty\int_{u_1}^\infty(1-e^{-u_1})^p(1-e^{-u_2})^q(u_1)^{n-1}(u_2-u_1)^{m-n-1}e^{-ku_2}du_2du_1.$$

By using the transformation $z=u_2-u_1$, and after some algebra, we get

$$I_{p,q}(n,m,k)=\sum_{i=0}^{p}\sum_{j=0}^{q}(-1)^{i+j}\binom{p}{i}\binom{q}{j}\frac{k^n}{(i+j+k)^n(j+k)^{m-n}},\ p,q=0,1,2,\ldots. \tag{3.1}$$

Now, by using (??), we get

$$f_{[n,m,k]}^{(u)}(y_1,y_2)=\int_0^\infty\int_{x_1}^\infty g_{m,n,k}^{(u)}(x_1,x_2)[\theta_2\exp(-\theta_2y_1)+3\alpha\theta_2\exp(-\theta_2y_1)(2(1-\exp(-\theta_1x_1))-1)$$
$$\times(2(1-\exp(-\theta_2y_1))-1)+\frac{5}{4}\alpha^2\theta_2\exp(-\theta_2y_1)(3(2(1-\exp(-\theta_1x_1))-1)^2-1)$$
$$\times(3(2(1-\exp(-\theta_2y_1))-1)^2-1)][\theta_2\exp(-\theta_2y_2)+3\alpha\theta_2\exp(-\theta_2y_2)$$
$$\times(2(1-\exp(-\theta_1x_2))-1)(2(1-\exp(-\theta_2y_2))-1)\frac{5}{4}\alpha^2\theta_2\exp(-\theta_2y_2)$$
$$\times(3(2(1-\exp(-\theta_1x_2))-1)^2-1)(3(2(1-\exp(-\theta_2y_2))-1)^2-1)]dx_2dx_1.$$

By using the relations $f_{V_1}=2f_YF_Y$ and $f_{V_2}=3f_YF_Y^2$ and carrying out some algebra, we get

$$f_{[n,m,k]}^{(u)}(y_1,y_2)=\int_0^\infty\int_{x_1}^\infty g_{m,n,k}^{(u)}(x_1,x_2)[\theta_2\exp(-\theta_2y_1)+3\alpha(2(1-\exp(-\theta_1x_1))-1)(2\theta_2\exp(-\theta_2y_1)$$
$$\times(1-\exp(-\theta_2y_1))-\theta_2\exp(-\theta_2y_1))+\frac{5}{4}\alpha^2(12(1-\exp(-\theta_1x))^2$$
$$-12(1-\exp(-\theta_1x-1))+2)(12\theta_2\exp(-\theta_2y_1)(1-\exp(-\theta_2y_1)^2)-12\theta_2\exp(-\theta_2y_1)$$
$$\times(1-\exp(-\theta_2y_1))+2\theta_2\exp(-\theta_2y_1))][\theta_2\exp(-\theta_2y_2)+3\alpha(2(1-\exp(-\theta_1x_2))-1)$$
$$\times(2\theta_2\exp(-\theta_2y_2)(1-\exp(-\theta_2y_2))-\theta_2\exp(-\theta_2y_2))+\frac{5}{4}\alpha^2(12(1-\exp(-\theta_1x_2))^2$$
$$-12(1-\exp(-\theta_1x_2))+2)(12\theta_2\exp(-\theta_2y_2)(1-\exp(-\theta_2y_2)^2)-12\theta_2\exp(-\theta_2y_2)$$
$$\times(1-\exp(-\theta_2y_2))+2\theta_2\exp(-\theta_2y_2))]dx_2dx_1.$$

On the other hand, upon using (??), with $p=1$ and $q=0$, for $t=1$, and with $p=0$ and $q=1$, for $t=2$, we get after some algebra

$$\int_0^\infty\int_{x_1}^\infty\frac{\alpha k^n}{\Gamma(n)\Gamma(m-n)}(2(1-\exp(-\theta_1x_t))-1)(-\log\exp(-\theta_1x_1))^{n-1}(\exp(-\theta_1x_2))^{k-1}\left(-\log\frac{\exp(-\theta_1x_2)}{\exp(-\theta_1x_1)}\right)^{m-n-1}$$

$$\frac{\theta_1^2\exp(-\theta_1x_1)\exp(-\theta_1x_2)}{\exp(-\theta_1x_1)}dx_2dx_1=\begin{cases}\alpha\left(2I_{1,0}(n,m,k)-1\right)=\alpha\left(1-2\left(\frac{k}{k+1}\right)^n\right)=\Lambda_{n,k:1}^{(\alpha)}, & t=1,\\ \alpha\left(2I_{0,1}(n,m,k)-1\right)=\alpha\left(1-2\left(\frac{k}{k+1}\right)^m\right)=\Lambda_{m,k:1}^{(\alpha)}, & t=2.\end{cases}$$

Similarly, we can obtain $\Lambda_{n,k:2}^{(\alpha)},\Lambda_{m,k:2}^{(\alpha)},\Lambda_{n,m,k:1}^{(\alpha)},\Lambda_{n,m,k:2}^{(\alpha)},\Lambda_{n,m,k:3}^{(\alpha)}$, and $\Lambda_{n,m,k:4}^{(\alpha)}$. This completes the proof.

Remark 3.1 The case $k = 1$, which mainly includes the case of record values, was handled by Husseiny et al. (2022)

$$f_{[n,m]}^{(u)}(y_1, y_2) = \theta_2 \exp(-\theta_2 y_1)\theta_2 \exp(-\theta_2 y_2) + \left(3\Lambda_{n:1}^{(\alpha)} - \frac{5}{2}\Lambda_{n:2}^{(\alpha)}\right)\theta_2 \exp(-\theta_2 y_2)(2\theta_2 \exp(-\theta_2 y_1)$$

$$\times (1 - \exp(-\theta_2 y_1)) - \theta_2 \exp(-\theta_2 y_1)) + \left(3\Lambda_{m:1}^{(\alpha)} - \frac{5}{2}\Lambda_{m:2}^{(\alpha)}\right)\theta_2 \exp(-\theta_2 y_1)$$

$$\times (2\theta_2 \exp(-\theta_2 y_2)(1 - \exp(-\theta_2 y_2)) - \theta_2 \exp(-\theta_2 y_2)) + 5\Lambda_{n:2}^{(\alpha)}\theta_2 \exp(-\theta_2 y_2)$$

$$\times (3\theta_2 \exp(-\theta_2 y_1)(1 - \exp(-\theta_2 y_1)^2) - 2\theta_2 \exp(-\theta_2 y_1)(1 - \exp(-\theta_2 y_1)))$$

$$+ 5\Lambda_{m:2}^{(\alpha)}\theta_2 \exp(-\theta_2 y_1)(3\theta_2 \exp(-\theta_2 y_2)(1 - \exp(-\theta_2 y_2)^2) - 2\theta_2 \exp(-\theta_2 y_2)$$

$$\times (1 - \exp(-\theta_2 y_2))) + \left(9\Lambda_{n,m:1}^{(\alpha)} - \frac{15}{2}\Lambda_{n,m:2}^{(\alpha)} - \frac{15}{2}\Lambda_{n,m:3}^{(\alpha)} + \frac{25}{4}\Lambda_{n,m:4}^{(\alpha)}\right)$$

$$\times (2\theta_2 \exp(-\theta_2 y_1)(1 - \exp(-\theta_2 y_1)) - \theta_2 \exp(-\theta_2 y_1))(2\theta_2 \exp(-\theta_2 y_2)(1 - \exp(-\theta_2 y_2))$$

$$-\theta_2 \exp(-\theta_2 y_2)) + \left(15\Lambda_{n,m:2}^{(\alpha)} - \frac{25}{2}\Lambda_{n,m:4}^{(\alpha)}\right)(3\theta_2 \exp(-\theta_2 y_1)(1 - \exp(-\theta_2 y_1)^2)$$

$$-2\theta_2 \exp(-\theta_2 y_1)(1 - \exp(-\theta_2 y_1)))(2\theta_2 \exp(-\theta_2 y_2)(1 - \exp(-\theta_2 y_2)) - \theta_2 \exp(-\theta_2 y_2))$$

$$+ \left(15\Lambda_{n,m:3}^{(\alpha)} - \frac{25}{2}\Lambda_{n,m:4}^{(\alpha)}\right)(2\theta_2 \exp(-\theta_2 y_1)(1 - \exp(-\theta_2 y_1))\theta_2 \exp(-\theta_2 y_1))$$

$$\times (3\theta_2 \exp(-\theta_2 y_2)(1 - \exp(-\theta_2 y_2)^2) - 2\theta_2 \exp(-\theta_2 y_2)(1 - \exp(-\theta_2 y_2))) + 25\Lambda_{n,m:4}^{(\alpha)}$$

$$\times (3\theta_2 \exp(-\theta_2 y_1)(1 - \exp(-\theta_2 y_1)^2) - 2\theta_2 \exp(-\theta_2 y_1)(1 - \exp(-\theta_2 y_1)))(3\theta_2 \exp(-\theta_2 y_2) \times (1 - \exp(-\theta_2 y_2)^2) - 2\theta_2 \exp(-\theta_2 y_2)(1 - \exp(-\theta_2 y_2))),$$

where $\Lambda_{m:1}^{(\alpha)}$ and $\Lambda_{m:2}^{(\alpha)}$ are defined via Theorem 2.1. by replacing n with m in $\Lambda_{n:1}^{(\alpha)}$ and $\Lambda_{n:2}^{(\alpha)}$, respectively,

$$\Lambda_{n,m:1}^{(\alpha)} = \alpha^2[2^{n-m+2} \times 3^{-n} - 2^{-n+1} - 2^{-m+1} + 1],$$

$$\Lambda_{n,m:2}^{(\alpha)} = \alpha^3[2 - 3 \times 2^{-n+2} + 4 \times 3^{-n+1} - 2^{-m+2} + 3^{-n+1} \times 2^{n-m+3} - 3 \times 2^{-n-m+3}],$$

$$\Lambda_{n,m:3}^{(\alpha)} = \alpha^3[2 - 3 \times 2^{-m+2} + 4 \times 3^{-m+1} - 2^{-n+2} + 3^{-n+1} \times 2^{n-m+3} - 3^{n-m+1} \times 2^{-2n+3}],$$

and

$$\Lambda_{n,m:4}^{(\alpha)} = \alpha^4[4 - 3 \times 2^{-m+3} - 3 \times 2^{-n+3} + 8 \times 3^{-m+1} + 8 \times 3^{-n+1} + 3^{-n+2} \times 2^{n-m+4}$$

$$-9 \times 2^{-n-m+4} - 3^{n-m+2} \times 2^{-2n+4} + 16 \times 3^{n-m+2} \times 5^{-n}].$$

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