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A novel stochastic technique using the dual phase-lag thermoelasticity model

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ABSTRACT : ***In this paper, a novel technique of stochastic thermoelastic interactions by using the theory of dual phase-lag is studied. A one-dimensional (1-D) problem was discussed. Silicon material was regarded as an example of our half space ($x \geq 0$) problem. As the nature is changeable, the boundary conditions were chosen to be also random by combining a random function to it. So, a noise was added to the problem to make it more actual. White noise was assumed to be the additional noise because it is the most prevalent type. The random function was regarded as the Wiener process function. Laplace transform was the method used to solve the problem in (1-D) numerically. The deterministic solutions were got numerically by Laplace method. The Wiener process was considered to get the stochastic solutions simultaneously. The variance of the distributions was considered and discussed graphically. A comparison between the deterministic and stochastic solutions were done graphically.***

KEYWORDS: Dual phase-lag; Laplace transform; stochastic distribution; Wiener process; white noise; deterministic solutions.

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I. INTRODUCTION

The standard thermoelasticity hypothesis, which was based on the original Fourier's law and was developed by Biot [1], predicts that heat can travel at any speed. From a physical standpoint, this phenomenon looks implausible. The weakness is highlighted since the theory includes a parabolic-type heat conduction equation. The results of experimental studies that demonstrate the presence of finite thermal wave speed cannot be explained by this hypothesis (see [2–6]). This inspired a lot of academics to focus on finding a solution to this problem. Numerous novel models based on a hyperbolic-type heat conduction equation were developed as a result of this research, which also included the limited heat transmission. It has been authenticated in numerous publications [7–15]. The Cattaneo-Vernotte heat conduction model [17–19] was modified by Lord and Shulman [16] to incorporate one relaxation time parameter and describe the restricted speed of the thermal signal. After this, Green and Lindsay [20] created a coupled theory of thermoelasticity using the explicit constitutive equations. In order to create new coupled thermoelastic theories, Green and Naghdi [21–23] added ν as a new constitutive variable, such that; ν represents the the movement of heat. Tzou [24, 25] later created the dual phase lag hypothesis. It was suggested by him that the macroscopic formulation take into account the microstructural implications of heat transport phenomena. He used two phase-lags to connect the heat flux vector $\vec{q}$ to the temperature gradient $\vec{\nabla}\theta$ in order to demonstrate the effects of microstructural changes on the heat transport process as:

$$\vec{q}\left(r,t+\tau_q\right)=-K\vec{\nabla}\theta\left(r,t+\tau_\theta\right)$$

Such that; τ_q in addition τ_q represent the phase lag parameters of the medium. Then, in a manner similar to dual phase-lag thermoelasticity theory, Roychoudhuri [26] provided an addition to the Green-Naghdi theory of the third type by incorporating an additional phase lag, τ_v, in the gradient of the thermal displacement. Recently, Quintanilla [27], changed how he approached the three-phase-lag models and examined the distant places and stability of the recently presented model by using $\tau_v > \tau_q = \tau_\theta$. Stochastic processes are mathematical tools that assist us in coping with the system's randomness. In essence, it is a set of random variables. Due to the fact that more options are taken into account when calculating than in the deterministic case, it makes the mathematical issue more real. It makes it easier to employ various samples so that the process can continue even in the absence of initial conditions. Other paths, nevertheless, might be more likely than some others [28–30]. There are several factors that call for switching from a strict deterministic model to a stochastic one, but two stand out [31]. First, there is a lack of complete system isolation. Second, because not all of the variables that make up the actual physical system may be included. Different thermal issues with randomized accessibility and issues in different mediums were studied by Ahmadi [32], Chen and Tien [33], Kellar et al. [34], and Tzou [35]. Considering the boundary conditions, Chiba, Ahmadi [36] and Sugano [37], and Gaikovich [38] have investigated the issues in greater detail. Consideration was given to the issues concerning stochastic internal heat generation by Val'kovskaya and Lenyuk [39]. In modified thermoelasticity and advanced thermoelastic diffusion, Sherief et al [40, 41], studied the stochastic thermal shock difficulties. The impacts of stochastic thermomechanical loads were then investigated by Kant and Mukhopadhyay [42, 43] concerning the theory of thermos-elasticity with no energy loss and the theory of thermos-elasticity with two relaxation coefficients.

II. Basic equations

Tzou [24], introduced the theory of dual phase-lag for heat conduction, so considering this theory we will research the coupling effects of thermoelastic interactions. With the help of Tzou [24, 25] and Chandrasekharaiah [8], the fundamental guiding equations with no body forces, neither heat sources are:

heat conduction with dual phase-lags theory:

$$\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^{\ 2}}{2}\frac{\partial^2}{\partial t^2}\right)q_i=-K\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\frac{\partial\theta}{\partial x} \tag{1}$$

Equation of energy:

$$-q_{i,j}=T_0\rho\frac{\partial s}{\partial t} \tag{2}$$

Equation of entropy:

$$T_0\rho S=\rho c_e\theta+T_0\alpha e_{kk} \tag{3}$$

Equation of motion:

$$\sigma_{ij,i}=\rho\ddot{u}_i \tag{4}$$

The relation between Stress–strain–temperature:

$$\sigma_{ij}=\lambda e_{kk}\delta_{ij}+2\mu e_{ij}-\alpha\theta\delta_{ih} \tag{5}$$

The relation between Strain–displacement:

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{6}$$

Such that; θ, is the increasing temperature, T_0, is the indication temperature, q_i is the vector of the heat flux, e_{ij} is the components of strain, σ_{ij}, is the stress component, u_i is the components of the displacement, entropy is S, the specific heat is c_e, the density is ρ, the Lame´s constants are (λ, μ) K, α, τ_θ *and* τ_q, are thermos-elasticity parameters, such that $\alpha = (3\lambda + 2\mu)\alpha_T$, where; α_T, denote the expansion coefficients, thermal conductivity is K, the phase-lag of the heat flux additionally, the temperature gradient's phase lag respectively.

1. Formulation of the problem

Let an elastic isotropic homogeneous media. For a half-space one-dimensional (1-D), problem. The variables must be constrained by boundaries at infinity of x . the main variables are; vector displacement $\vec{u} = \left(u(x,t), 0, 0\right)$, and temperature distribution. We obtain the governing equations as:

$$K\left(1 + \tau_\theta \frac{\partial}{\partial t}\right)\frac{\partial^2 \theta}{\partial x^2} = \left(1 + \tau_q \frac{\partial}{\partial t} + \frac{\tau_q^{\ 2}}{2}\frac{\partial^2}{\partial t^2}\right)\left(\rho c_e \frac{\partial \theta}{\partial t} + \alpha T_0 \frac{\partial^2 u}{\partial t \partial x}\right) \tag{7}$$

The motion equation can be rewritten as:

$$\rho \frac{\partial^2 u}{\partial t^2} = \left(\lambda + 2\mu\right)\frac{\partial^2 u}{\partial x^2}\alpha \frac{\partial \theta}{\partial x} \tag{8}$$

The constitutive relation

$$\sigma_{xx} = \left(\lambda + 2\mu\right)\frac{\partial u}{\partial x} - \alpha\theta \tag{9}$$

To simplify the problem, we put the dimensionless form to the variables as:

$$x' = C_1 x\,\xi x\,,\ t' = C_1^{\ 2}\xi t,\ \theta' = \frac{\theta}{T_0},\ u' = \frac{C_1\left(\lambda + 2\mu\right)}{\alpha T_0},$$

$$\tau'_q = C_1^{\ 2}\xi\tau_q,\ \tau'_\theta = C_1^{\ 2}\xi\tau_\theta,\ \varepsilon = \frac{\alpha^2 T_0}{\rho^2 c_e C_1^{\ 2}} \text{ and } \sigma'_{xx} = \frac{\sigma_{xx}}{T_0\alpha};$$

Such that; the speed of thermal wave is C_1, and is defined as $C_1 = \sqrt{\frac{(\lambda + 2\mu)}{\rho}}$, ξ is the thermoelasticity coupling constant, and equals $\frac{c_e \rho}{K}$.

Then we can transform the basic equations (7)-(9), by the help of the above dimensionless variables;

$$\left(1+\tau_q \frac{\partial}{\partial t}\right)\frac{\partial^2\theta}{\partial x^2}=\left(1+\tau_q \frac{\partial}{\partial t}+\frac{\tau^2_q}{2}\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial\theta}{\partial t}+\varepsilon\frac{\partial^2 u}{\partial t\,\partial x}\right) \tag{10}$$

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}-\frac{\partial\theta}{\partial x} \tag{11}$$

$$\sigma_{xx}=\frac{\partial u}{\partial x}-\theta \tag{12}$$

For more simplification for the problem; we neglected the dash.

III. The problem solution using Laplace method

By using Laplace technique to equations (10)-(12), we get;

$$\left(D^2-\alpha_1\right)\bar{\theta}(s)=\varepsilon\alpha_1 D\bar{u}(s) \tag{13}$$

$$\left(D^2-s^2\right)\bar{u}(s)=D\bar{\theta}(s) \tag{14}$$

$$\sigma_{xx}(s)=D\bar{u}(s)-\bar{\theta}(s) \tag{15}$$

Such that;

$$D=\frac{d}{dx},\quad \alpha_1=\frac{a}{b}. \tag{16}$$

where; $a=s\left(1+\tau_q s+\frac{1}{2}\tau_q{}^2 s^2\right),\ b=1+\tau_q s$

The boundary conditions in Laplace form are;

$$\bar{\theta}(x,s)=\bar{\theta}_0(x,s),\qquad \bar{\sigma}_{xx}(x,s)=0. \tag{17}$$

To get the solution of the equations (13)-(15), we use the elimination method, we get the auxiliary equation as;

$$D^4-\Pi_1 D^2+\Pi_2=0 \tag{18}$$

$$\left.\begin{aligned}\Pi_1&=s^2+\alpha_1(\varepsilon+1)\\ \Pi_2&=s^2\alpha_1\end{aligned}\right\} \tag{19}$$

In the case of linearity problem;

$$\bar{\theta}(x,s)=\sum_{i=1}^{2}\bar{M}_i(s)\exp(-k_i x) \tag{20}$$

In a similar way, the other fields solution is;

$$\bar{u}(x,s)=\sum_{i=1}^{2}\bar{M}_i'(s)\exp(-k_i x)=\sum_{i=1}^{2}H_{1i}\bar{M}_i(s)\exp(-k_i x) \tag{21}$$

And

$$\bar{\sigma}_{xx}(x,s)=\sum_{i=1}^{2}\bar{M}_i''(s)\exp(-k_i x)=\sum_{i=1}^{2}H_{2i}\bar{M}_i(s)\exp(-k_i x) \tag{22}$$

Where; M_i, M_i' and M_i'' are unknown functions of s only, and

$$\left.\begin{aligned} H_{1i} &= \frac{k_i^{\,2}-\alpha_1}{k_1\alpha_1} \\ H_{2i} &= -1+H_{1i}k_i \end{aligned}\right\}. \tag{23}$$

IV. Temperature distribution (deterministic and stochastic)

V. Deterministic Temperature

Equation (20) may be expanded as:

$$\bar{\theta}(x,s)=M_1\exp(-k_1x)+M_2\exp(-k_2x). \tag{24}$$

Defining the boundary condition of temperature distribution is

$$\theta_0(t)=h(t)\theta^*. \qquad \text{when } t>0 \tag{25}$$

Such that $h(t)$ is the unit step function, θ^* is a constant.

Taking Laplace transfer for the two sides of the aforementioned equation, yield;

$$\bar{\theta}_0(t)=\frac{\theta^*}{s}. \tag{26}$$

5.1. Stochastic Temperature

Considering the boundary in equation (25), defined as:

$$\theta_0(t)=\theta(t)+\varphi_0(t) \tag{27}$$

Where $\varphi_0(t)$ is stochastic process depending upon t, satisfying

$$E\left[\varphi_0(t)\right]=0 \tag{28}$$

A random function $Y(t)$, as [40,44] fulfil the next property;

$$E[L\{Y(t)\}]=L[E\{Y(t)\}] \tag{29}$$

Since each physical quantity contains a boundary condition, then adding the random function converts it to a stochastic one. Then we have from equation (24), that;

$$E\left[\bar{T}(x,s)\right]=L\left[E\{T(x,t)\}\right]=\bar{T}(x,s) \tag{30}$$

It's observed that the temperature's mean equals its deterministic case.

So, equation (24) can be transformed into;

$$\bar{\theta}(x,s)=\bar{\Theta}(x,s)\bar{\theta}_0(s) \tag{31}$$

Where $\bar{\Theta}(x,s)$, is defined as;

$$\bar{\Theta}=\left(\frac{h21}{h21-h22}\exp(-k_2x)-\frac{E1h22}{h21-h22}\exp(-k_1x)\right) \tag{32}$$

Now, using the boundary condition, defined in equation (27), we get;

$$\bar{\theta}(x,s)=\bar{\Theta}(x,s)\left(\bar{\theta}(s)+\bar{\varphi}_0(s)\right) \tag{33}$$

Applying the convolution property to get the inverse of Laplace transform, we get;

$$\theta(x,t)=\theta^1(x,t)+\int_0^t\Psi(r)\Theta(x,t-r)du \tag{34}$$

Where $\Psi(r)$ is the white noise function, deterministic temperature is $\theta^1(x,t)$ and $\Theta(x,t-r)$ is the inverse laplace of eq (32).

Hence, equation (34), may be rearranged as:

$$\theta(x,t)=\theta^1(x,t)+\int_0^t\Theta(x,t-r)dW(r) \tag{35}$$

Such that, $dW(r)$ defines the Wiener process.

Now, the variance can be got by squaring equation (34), we get;

$$\left(\theta(x,t)\right)^2=\left(\theta^1(x,t)\right)^2+\int_0^t\int_0^t\Psi(r_1)\Psi(r_2)\Theta(x,t-r_1)\Theta(x,t-r_2)+$$
$$2\int_0^t\theta^1(x,t)\Psi(r)\Theta(x,t-r) \tag{36}$$

Introducing the expectation operator to two sides of the equation above, we have;

$$E\left[\theta(x,t)\right]^2=\left(E\left[\theta^1(x,t)\right]\right)^2+\int_0^t\int_0^t E\left[\Psi(r_1)\Psi(r_2)\right]\Theta(x,t-r_1)\Theta(x,t-r_2)$$
$$+2\int_0^t\theta^1(x,t)E\left[\Psi(r)\right]\Theta(x,t-r) \tag{37}$$

Where, $E\left[\Psi(r)\right]=0$ and $E\left[\Psi(r_1)\Psi(r_2)\right]=\delta(r_1-r_2)$. (38)

Hence

$$E\left[\theta(x,t)\right]^2-\left(E\left[\theta^1(x,t)\right]\right)^2=\int_0^t\int_0^t\Theta(x,t-r_1)\Theta(x,t-r_2)\delta(r_1-r_2)dr_1dr_2 \tag{39}$$

Using the following property;

$$\int_a^b f(x)f(x-x_0)dx=f(x_0),\qquad a<x_0<b. \tag{40}$$

Hence,

$$Var\left[\theta(x,t)\right]=\int_0^t\Theta(x,t-r_1)\Theta(x,t-r_2)dr_1 \tag{41}$$

Putting $r_1=r_2$, we get;

$$Var\left[\theta(x,t)\right]=\int_0^t\left[\Theta(x,t-r_1)\right]^2dr_1. \tag{42}$$

Replacing $t-r_1$ by ϑ, we get;

$$Var\left[\theta(x,t)\right]=-\int_t^o\left[\Theta(x,\vartheta)\right]^2d\vartheta=\int_0^t\left[\Theta(x,\vartheta)\right]^2d\vartheta. \tag{43}$$

VI. Stress distribution

VII. Deterministic stress

Equation (22) can be expanded as;

$$\bar{\sigma}(x,s)=H_{21}M_1\exp(-k_1x)+H_{22}M_2\exp(-k_2x). \tag{44}$$

6.1. Stochastic stress

Recalling the boundary condition of stochastic process which at equation (27), and performing the same method as the above section, we get;

$$E\left[\bar{\sigma}(x,s)\right]=L\left[E\left\{\sigma(x,t)\right\}\right]=\bar{\sigma}(x,s). \tag{45}$$

Again, it’s observed that, temperature’s mean equals its deterministic case. Equation (44), is rearranged as;

$$\bar{\sigma}(x,s)=\bar{\Omega}(x,s)\bar{\theta}_0(x,s) \tag{46}$$

Where
$$\bar{\Omega}(x,s)=\left(\frac{H22H21}{H21-H22}\exp(-k_2x)-\frac{H21H22}{H21-H22}\exp(-k_1x)\right) \tag{47}$$

After using the condition of equation (27) we get;

$$\bar{\sigma}(x,s)=\bar{\Omega}(x,s)\left(\bar{\theta}(s)+\bar{\varphi}_0(s)\right) \tag{48}$$

After cancellation and using the convolution property of Laplace transform of the above equation we get,

$$\sigma(x,t)=\sigma^1(x,t)+\int_0^t\Omega(x,t-r)dW(r) \tag{49}$$

Where the deterministic stress is $\sigma^1(x,t)$ and the inverse Laplace of eq (48) is $\Omega(x,t-r)$. Using the same technique used above, the stress distribution variance can be got as:

$$Var\left[\sigma(x,t)\right]=\int_0^t\Omega^2(x,\vartheta)d\vartheta \tag{50}$$

VIII. Displacement distribution

IX. Deterministic displacement

Equation (21) is expanded as;

$$\bar{u}(x,s)=H_{11}M_1\exp(-k_1x)+H_{12}M_2\exp(-k_2x). \tag{51}$$

X. Stochastic displacement

Recalling the boundary condition of stochastic process which at equation (27), and performing the same method as the above section, we get;

$$E\left[\bar{u}(x,s)\right]=L\left[E\left\{u(x,t)\right\}\right]=\bar{u}(x,s). \tag{45}$$

Again, it’s observed that, the temperature’s mean equals its deterministic case. Equation (44), can be rearranged as;

$$\bar{u}(x,s)=\bar{U}(x,s)\bar{\theta}_0(x,s) \tag{46}$$

Such that
$$\bar{U}(x,s)=\left(\frac{H12H21}{H21-H22}\exp(-k_2x)-\frac{H11H22}{H21-H22}\exp(-k_1x)\right) \tag{47}$$

After using the condition of equation (27) we get;

$$\bar{u}(x,s)=\bar{U}(x,s)\left(\bar{\theta}(s)+\bar{\varphi}_0(s)\right) \tag{48}$$

After cancellation and using the convolution property of Laplace transform of the above equation we get,

$$u(x,t)=u^1(x,t)+\int_0^t U(x,t-r)\,dW(r) \tag{49}$$

Where $u^1(x,t)$ define the deterministic stress. $U(x,t-r)$ define the inverse Laplace of eq (48). Using the same technique used above, the variance can be got as:

$$Var\left[u(x,t)\right]=\int_0^t \Omega^2(x,\vartheta)\,d\vartheta \tag{50}$$

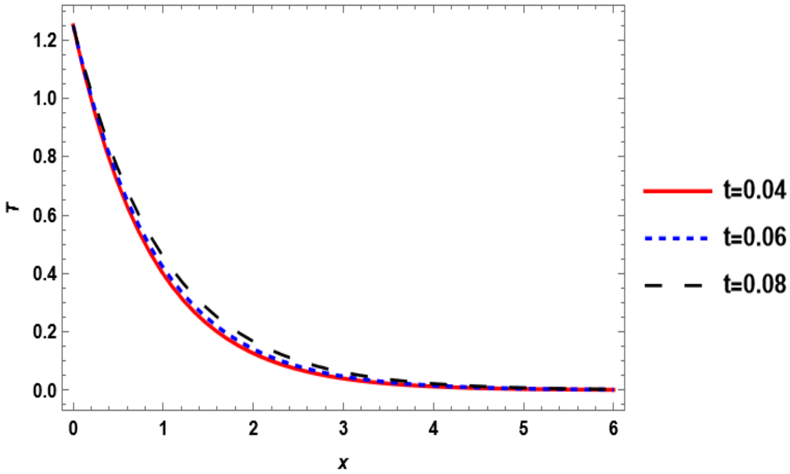
XI. Numerical results

Taking Silicon material as a semiconductor example of our problem, with the following constants [44-46].

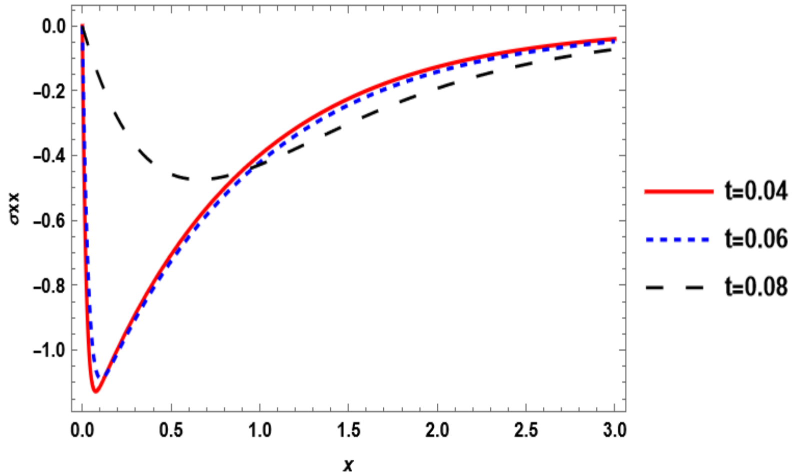
Unit	Symbol	Si
N/m^2	λ μ	6.4×10^{10} 6.5×10^{10}
kg/m^3	ρ	2330
K	T_0	800
$\sec(s)$	τ	$5\text{x}10^{-5}$
m^2/s	D_e	2.5×10^{-3}
m^3	d_n	$-9\text{x}10^{-31}$
eV	E_g	1.11
K^{-1}	α_t	$4.14\text{x}10^{-6}$
$Wm^{-1}K^{-1}$	k	150
$J/(kg\,K)$	C_e	695
m/s	$\tilde{s}$	2

Tacking the values of the dimensionaless incomes as; $\tau_\theta=0.01,\ \tau_q=0.015,\ t=.04$, $t_c=0.1,$ and $\theta^*=1$. By simulating computation numerically, the solutions were got graphically. The stochastic integration that follows Higham [47] is calculated using either the Brownian motion theory or a typical Wiener process. To distinguish both deterministic and stochastic distributions, a set of three sample paths were taken while computing all physical variables for stochastic distributions. Figure 1 (1a to 1c), display the deterministic temperature, stress and displacement, respectively, versus the distance for various values of times, namely; $t=0.04,\ t=0.06$ and $t=0.08$ in the case of thermoelasticity theory. Whereas, the figure 2 (2a to 2c), display the stochastic temperature, stress and displacement, respectively, versus the distance for specific value of time namely, $t=0.04$ in the case of thermoelasticity theory. Also figure 3 (3a to 3c), show a comparison between the deterministic and stochastic distributions of temperature, stress, and displacement against the distance when $t = 0.04$ in the context of thermoelasticity theory. Finally, figures 4 (4a to 4c), show the variance distribution of temperature, stress, and displacement against the distance at different values of time, namely; $t = 0.04,\ t = 0.06$ and $t = 0.08$ in the context of thermoelasticity theory. Figure 1 shows the difference in deterministic solutions in different values of times. We observe that they behave the same as a wave, but differ in the magnitude with a small amount. We note that the stochastic distributions in figure 2, begin very strong, then decrease till coincides with x-axis matching the conditions at infinity. Figure 3, it's notable that, the deterministic temperature distribution differs from the stochastic temperature distribution with small magnitudes over several sample paths before they eventually correspond fully. Figure 4, displays the variance distributions, which achieve the boundary and the initial conditions of the waves; we note that, the temperature and displacement distributions begins from the peak, then drop sharply to the minimum values till coincide with the zero line. The variance of stress distributions begins from zero then rise up to the maximum then drops again to the minimum till coincide at infinity. a stochastic solution has been developed by including the white noise in the boundary conditions.

Fig(1-a)



Fig(1-b)



Fig(1-c)

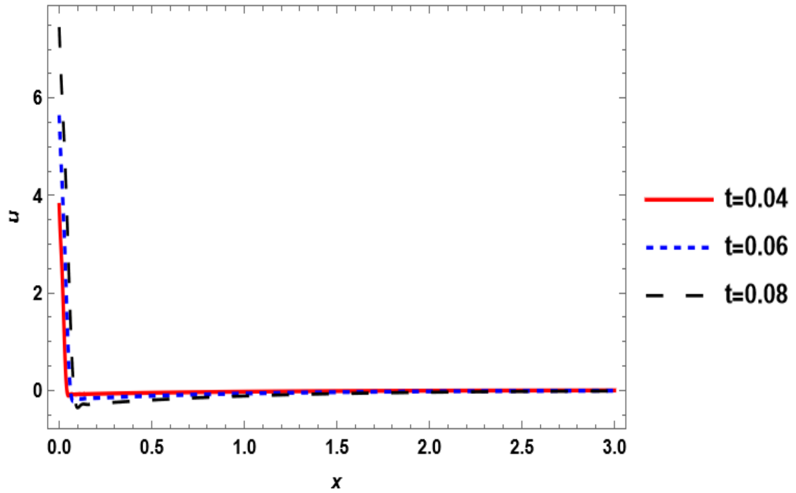
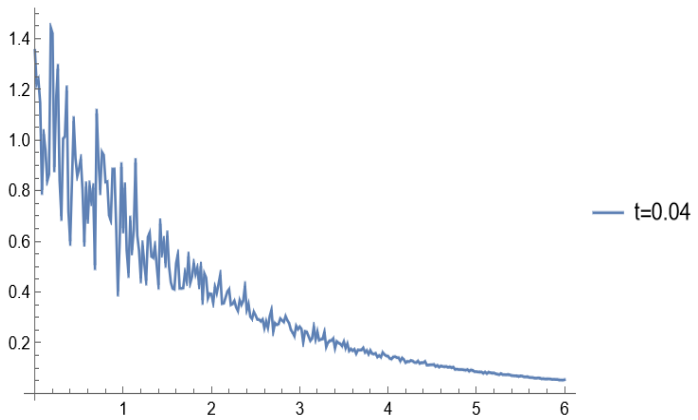
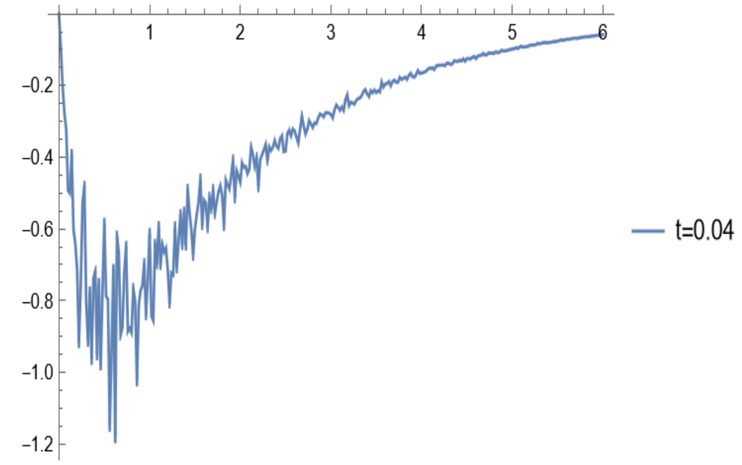


Figure 1: The deterministic of (temperature, stress and displacement distributions) versus the x-axis for various values of times at $t = 0.04,\ t = 0.06$ and $t = 0.08$.

Fig(2-a)



Fig(2-b)



Fig(2-c)

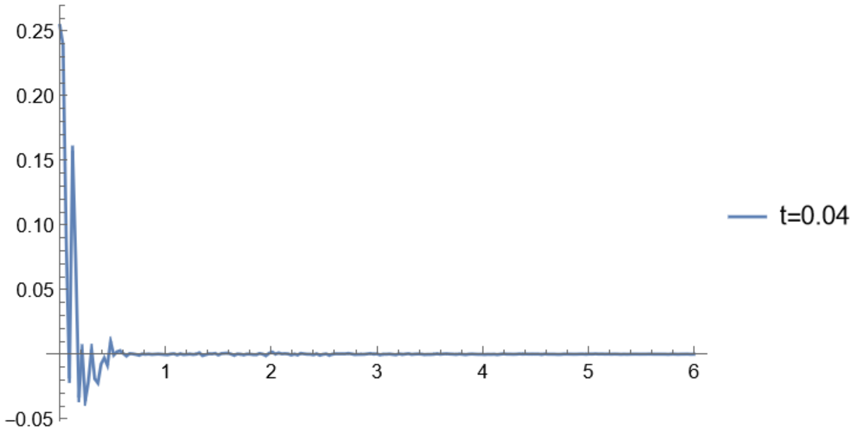
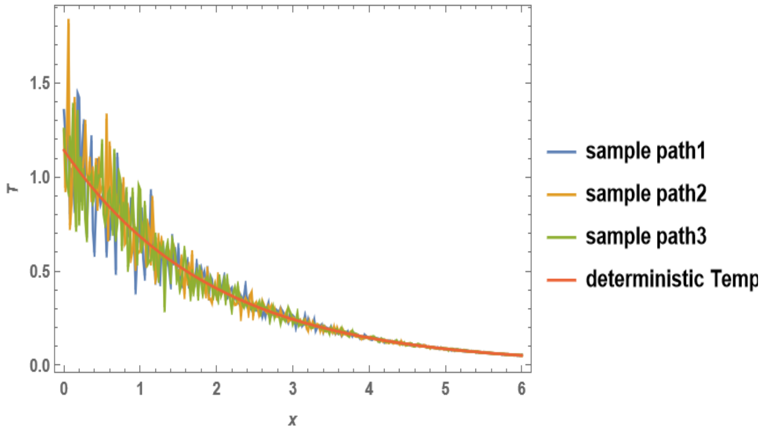
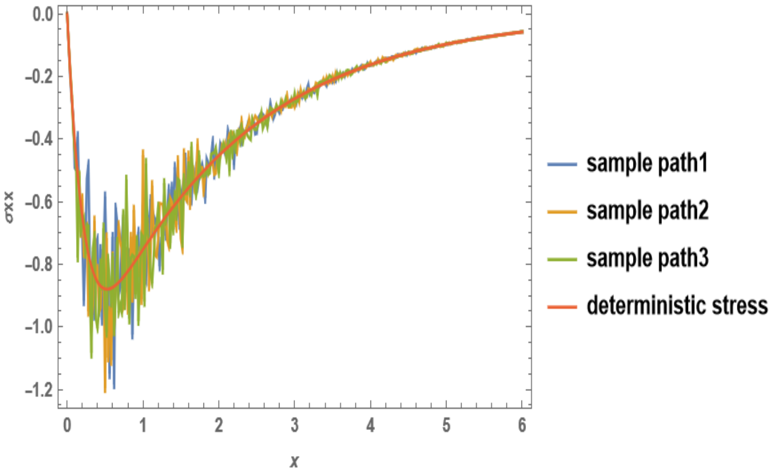


Fig2: The stochastic temperature, stress and displacement with x-axis at $t = 0.04$

Fig(3-a)



Fig(3-b)



Fig(3-c)

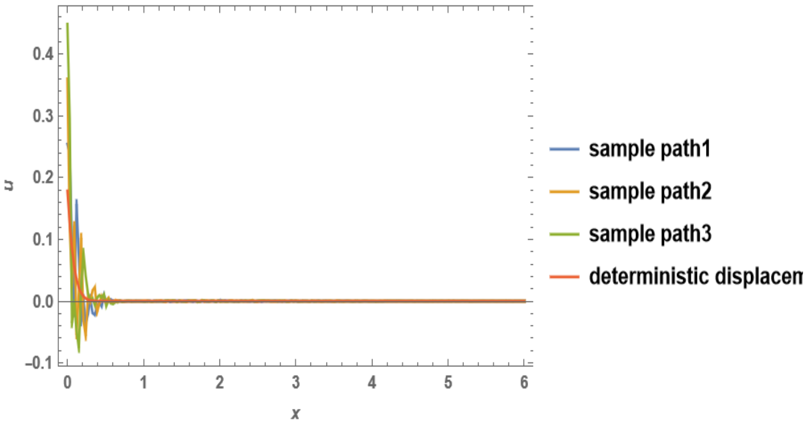
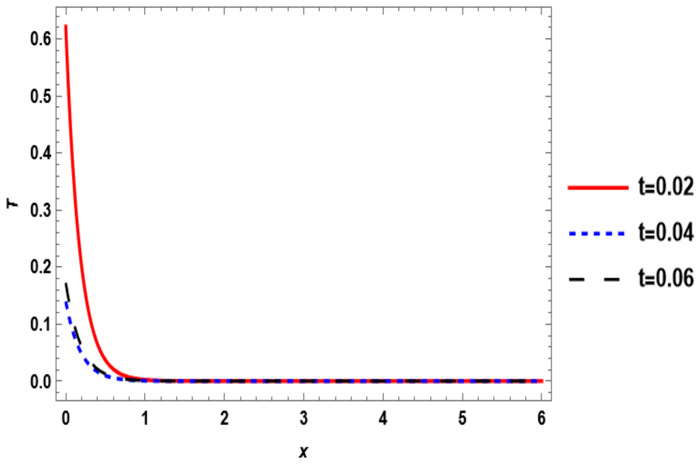
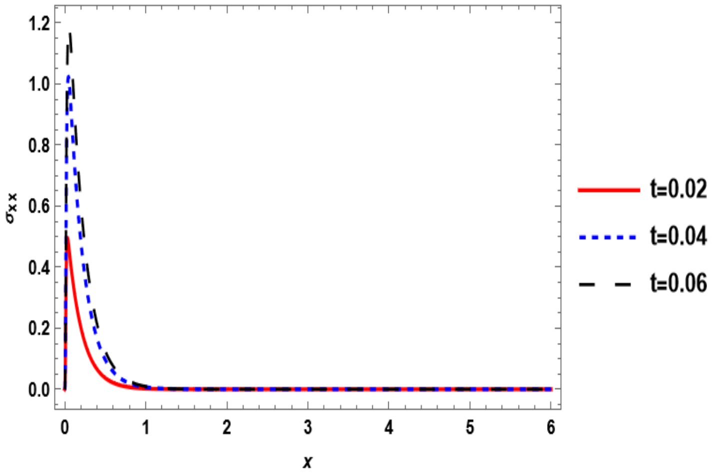


Figure (3(a-c)): The comparison between stochastic and deterministic distributions versus x-axis for various values of times at $t = 0.04,\ t = 0.06$ and $t = 0.08$.

Fig(4-a)



Fig(4-b)



Fig(4-c)

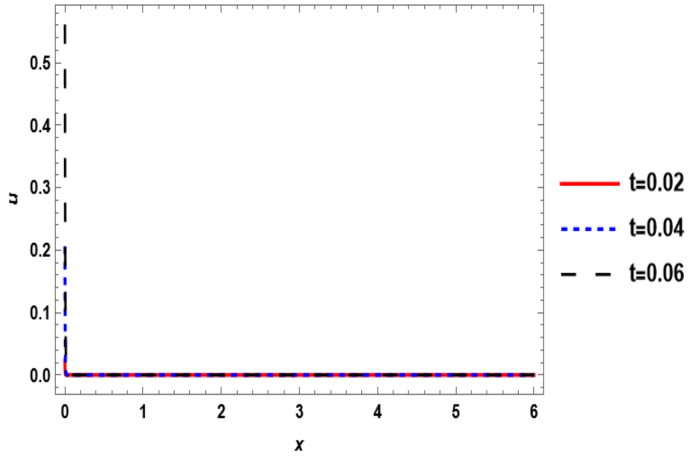


Figure 4: The variance of (temperature, stress and displacement) versus the distance for various values of times, namely; $t = 0.04,\ t = 0.06$ and $t = 0.08$.

XII. Conclusion

The one-dimensional half-space problem of dual phase lag in an isotropic elastic homogeneous material has been solved and explained. The problem was undergone a numerical analysis. Utilizing three sample paths and their means, a comparison between deterministic distributions and their equivalent stochastic distribution has been made. All variables disappear after a certain distance in both the case of a deterministic distribution and a stochastic distribution, demonstrating the existence of a finite zone of effects. The variance was found to be proportional to the squared of the intensity of the noise. The agreement between the mean of the stochastic solution and its corresponding deterministic solution for all physical fields has been validated by numerical reasoning. This synchronicity demonstrates the accuracy of the findings.

REFERENCES

[1] Biot, MA. Thermoelasticity and irreversible thermodynamics. J Appl Phys 1956; 27(3): 240–253.

[2] Ackerman, CC, and Guyer, RA. Temperature pulses in dielectric solids. Ann Phys 1968; 50(1): 128–185.

[3] Jackson, HE, Walker, CT, and McNelly, TF. Second sound in NaF. Phys Rev Lett 1970; 25(1): 26.

[4] Jackson, HE, and Walker, CT. Thermal conductivity, second sound, and phonon–phonon interactions in NaF. Phys Rev B 1971; 3(4): 1428.

[5] Rogers, SJ. Transport of heat and approach to second sound in some isotopically pure alkali–halide crystals. Phys Rev B 1971; 3(4): 1440.

[6] Narayanamurti, V, and Dynes, RC. Observation of second sound in bismuth. Phys Rev Lett 1972; 28(22): 1461.

[7] Chandrasekharaiah, DS. Thermoelasticity with second sound: a review. Appl Mech Rev 1986; 39(3): 355–376.

[8] Chandrasekharaiah, DS. Hyperbolic thermoelasticity: a review of recent literature. Appl Mech Rev 1998; 51(12): 705–729.

[9] Joseph, DD, and Preziosi, L. Heat waves. Rev Mod Phys 1989; 61(1): 41–73.

[10] Hetnarski, RB, and Ignaczak, J. Generalized thermoelasticity. J Therm Stresses 1999; 22(4–5): 451–476.

[11] Dreyer, W, and Struchtrup, H. Heat pulse experiments revisited. Continuum Mech Thermodyn 1993; 5(1): 3–50.

[12] Ozisik, MN, and Tzou, DY. On the wave theory in heat conduction. J Heat Transfer 1994; 116(3): 526–535.

[13] Ignaczak, J, and Ostoja-Starzewski, M. Thermoelasticity with finite wave speeds. Oxford: Oxford University Press, 2010.

[14] Me´ller, I, and Ruggeri, T. Extended thermodynamics (Springer Tracts on Natural Philosophy, vol. 37). New York: Springer, 1993.

[15] Mare´n, E. Does Fourier's law of heat conduction contradict the theory of relativity? Lat Am J Phys Educ 2011; 5(2): 402–405.

[16] Lord, HW, and Shulman, Y. A generalized dynamical theory of thermoelasticity. J Mech Phys Solids 1967; 15(5): 299–309.

[17] Cattaneo, C. A form of heat-conduction equations which eliminates the paradox of instantaneous propagation. Compt Rend 1958; 247: 431–433. [18] Vernotte, P. Les paradoxes de la the´orie continue de l'e´quation de lachaleur. Compt Rend 1958; 246: 3154–3155.

[19] Vernotte, P. Some possible complications in the phenomena of thermal conduction. Compt Rend 1961; 252: 2190–2191.

[20] Green, AE, and Lindsay, KA. Thermoelasticity. J Elast 1972; 2(1): 1–7. Gupta and Mukhopadhyay 19

[21] Green, AE, and Naghdi, PM. A re-examination of the basic postulates of thermomechanics. In: Proceedings of the Royal Society A 1991; 432: 171–194.

[22] Green, AE, and Naghdi, PM. On undamped heat waves in an elastic solid. J Therm Stresses 1992; 15(2): 253–264.

[23] Green, AE, and Naghdi, PM. Thermoelasticity without energy dissipation. J Elast 1993; 31(3): 189–208.

[24] Tzou, DY. A unified field approach for heat conduction from macro-to micro-scales. J Heat Transfer 1995; 117(1): 8–16.

[25] Tzou, DY. The generalized lagging response in small-scale and high-rate heating. Int J Heat Mass Transfer 1995; 38(17): 3231–3240.

[26] Roychoudhuri, SK. On a thermoelastic three-phase-lag model. J Therm Stresses 2007; 30(3): 231–238.

[27] Quintanilla, R. Some solutions for a family of exact phase-lag heat conduction problems. Mech Res Commun 2011; 38(5): 355–360.

[28] Hoel, PG, Port, SC, and Stone, CJ. Introduction to stochastic processes. Waveland Press, 1986.

[29] Platen, E, and Kloeden, P. Numerical Solution of Stochastic Differential Equations. Boca Raton, FL: CRC Press, 2006.

[30] Lawler, GF. Introduction to stochastic processes. Boca Raton, FL: CRC Press, 2006.

[31] Bellomo, N, and Flandoli, F. Stochastic partial differential equations in continuum physics—on the foundations of the stochastic interpolation method for ITO's type equations. Math Comput Sim 1989; 31(1–2): 3–17.

[32] Ahmadi, G. A perturbation method for studying heat conduction in solid with random conductivity. J Appl Mech 1978; 45: 933.

[33] Chen, YM, and Tien, CL. Penetration of temperature waves in a random medium. Stud Appl Math 1967; 46(1–4): 188–194.

[34] Keller, JB, Papanicolaou, GC, and Weilenmann, J. Heat conduction in a one-dimensional random medium. Commun Pure Appl Math 1978; 31(5): 583–592.

[35] Tzou, DY. Stochastic analysis of temperature distribution in a solid with random heat conductivity. J Heat Transfer 1988; 110(1): 23–29.

[36] Ahmadi, G. Heat conduction in solids with random initial conditions. J Heat Transfer 1974; 96(4): 474–477.

[37] Chiba, R, and Sugano, Y. Stochastic thermoelastic problem of a functionally graded plate under random temperature load. Arch Appl Mech 2007; 77(4): 215.

[38] Gaikovich, KP. Stochastic theory of temperature distribution and thermal emission of half-space with random time dependent surface temperature. IEEE Trans Geosci Remote Sens 1996; 34(2): 582–587.

[39] Val'kovskaya, VI, and Lenyuk, MP. Stochastic nonstationary temperature fields in a solid circular–cylindrical two-layer plate. J Math Sci 1996; 79(6): 1483–1487.

[40] Sherief, HH, El-Maghraby, NM, and Allam, AA. Stochastic thermal shock problem in generalized thermoelasticity. Appl Math Model 2013; 37(3): 762–775.

[41] Sherief, HH, El-Maghraby, NM, and Allam, AA. Stochastic thermal shock problem and study of wave propagation in the theory of generalized thermoelastic diffusion. Math Mech Solids 2017; 22(9): 1767–1789.

[42] Kant, S, and Mukhopadhyay, S. Investigation of a problem of an elastic half space subjected to stochastic temperature distribution at the boundary. Appl Math Model 2017; 46: 492–518

[43] Kant, S, and Mukhopadhyay, S. Investigation on effects of stochastic loading at the boundary under thermoelasticity with two relaxation parameters. Appl Math Model 2018; 54: 648–669.

[44] A. Hobiny, I. Abbas, Analytical solutions of photo-thermo-elastic waves in a non-homogenous semiconducting material, Res., Phys., 10, 385-390, (2018).

[45] I. Abbas and M. Marin, Analytical Solutions of a Two-Dimensional Generalized Thermoelastic Diffusions Problem Due to Laser Pulse, Iran. J. Sci. Technol.-Trans. Mech. Eng., 42(1), 57-71 (2018).

[46] M. Marin, M. Othman, A. Seadawy and C. Carstea, A domain of influence in the Moore-Gibson-Thompson theory of dipolar bodies, J. Taibah Uni. Sci., 14(1), 653-660 (2020).

[47] Higham, DJ. An algorithmic introduction to numerical simulation of stochastic differential equations. SIAM Rev 2001; 43(3): 525–546.