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Partial 2-Metric Spaces for Cyclic (ψ,φ,A,B)-Contractions in Fixed Point Theorems

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ABSTRACT **: There are a lot of generalizations of the concept of metric space . In this paper we using some fixed point theorems in partial 2- metric spaces and given an examples in it . And We propose the idea of Cyclic (ψ,φ,A,B)- Contraction in partial 2- metric spaces and investigate fixed point theorems for mappings meeting cyclical generalised contractive requirements in full partial 2-metric spaces in this research . The goal of this paper is to present some fixed point theorems for a mapping meeting a cyclical generalised contractive condition in partial 2- metric spaces based on a pair of altering distance functions . We establish that these mappings necessarily have unique fixed points in complete 2- metric spaces . Fixed point theorems for some contractions from this class are introduced and illustrative examples are given . Our results in that paper also generalises an existing result in 2- metric spaces .**

KEYWORDS: Partial metric spaces, contractive conditions, partial 2-metric spaces, a fixed point, cyclic map

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I. INTRODUCTION

Several generalizations of the metric spaces originated from a fixed point theory in diverse domains. In [3-4,6,13], a number of writers establish several metric space on fixed point theorems. Matthews [12] was one of them, and in 1994, he defined a normal metric spaces are generalised to partial metric spaces. Lateef [10] used contractive to prove fixed point theorems in 2-metric spaces. Abu Donia et al. also proposed an idea of partial 2-metric spaces in [5]. We'll try going over some of the most well-known partial metric space [1-2,7-9] and partial 2-metric space definitions and characteristics.

Definition [11]. Let X be a nonempty set. The mapping $p: X \times X \to [0,\infty)$ is said to be a partial metric on X if the following conditions are true. For any $x, y, z \in X$, we have,

(PM-1) $p(x,x) = p(y,y) = p(x,y)$ *if and only if* $x = y$,

(PM-2) $p(x,x) \le p(x,y)$,

(PM-3) $p(y,x) = p(x,y)$,

(PM-4) $p(x,z) \le p(x,y) + p(y,z) - p(y,y)$, and then the pair (X,ρ) is called a partial metric space, (for short PMS).

Definition [5]. A mapping $\rho: X^3 \to \mathbb{R}^+$ where X is a non-empty set, is said to be a partial 2-metric on X if the following conditions are holds true. For every x, y, z *and* u *in* we have,

(P2M-1) $\rho(x,x,x) = \rho(y,y,y) = \rho(z,z,z) = \rho(x,y,z)$ when at least two of x,y and z are equals,

(P2M-2) $\rho(x,x,x) \le \rho(x,y,z)$,

(P2M-3) $\rho(x,y,z)=\rho(x,z,y)=\rho(z,y,x)$,

(P2M-4) $\rho(x,y,z)\le\rho(x,y,u)+\rho(x,u,z)+\rho(u,y,z)-\rho(u,u,u)$. Then the pair (X,ρ) is called a partial 2-metric space; for short we write P2M space.

Example [5]. Let $x=\{0,1\}$ and let $\rho(x,y,z)=\begin{cases}2 & \text{if } x=y=z=0)\\ 1 & \text{otherwise}\end{cases}$, then (X,ρ) is a P2M space.

Theorem[5]. Let (X,ρ) be a partial 2-metric space, and suppose the function

$d_\rho : X^3 \longrightarrow [0,\infty)$, denoted by

$d_\rho(x,y,z)=3\rho(x,y,z)-\rho(x,x,x)-\rho(y,y,y)-\rho(z,z,z)$, then (X, d_ρ) is a 2-metric space.

Definition [5]. A sequence $\{x_n\}$ in a partial 2-metric space (X,ρ) converges to a point x in X if $\lim_{n\to\infty}\rho(x_n,x,z)=\rho(x,x,x)$ for all z in X.

Theorem [5]. A sequence $\{x_n\}$ in a partial 2-metric space (X,ρ), converges to a point x in X if $\lim_{n\to\infty}\rho(x_n,x,z)=\rho(x,x,x)$ for all z in X.

Definition [5]. A sequence $\{x_n\}$ in a partial 2-metric space (X,ρ), is said to be a Cauchy sequence if the $lim_{n,m\to\infty}\rho(x_n,x_m,z)$ exists and finite, for all z in X.

Definition [5]. Let (X,ρ) a partial 2-metric space is said to be complete if every Cauchy sequence in X converges to an element x in X such that $lim_{n,m\to\infty}\rho(x_n,x_m,z)=\rho(x,x,x)$, for all z in X.

Lemma [5]. (1) A sequence $\{x_n\}$ is a Cauchy sequence in a partial 2-metric space (X,p) if and only if $\{x_n\}$ is also a Cauchy sequence in the 2-metric space (X,d_ρ)

(2) (X,p) is complete if and only if (X,d_ρ) is also complete. Moreover

$\lim_{n\to\infty}\rho(x,x_n,z)=\lim_{n,m\to\infty}\rho(x_n,x_m,z)=\rho(x,x,x)\Leftrightarrow\lim_{n\to\infty}d_\rho(x,x_n,z)=0$.

Theorem [14]. Set A and B be nonempty closed subsets of a complete metric space (X,d). Assume that $T:A\cup B\to A\cup B$ is a cyclic map such that $d(Tx,Ty)\le kd(x,y)$ $\forall x\in A,\ y\in B.$ If $k\in[0,1)$, then T has a unique fixed point in A∩B.

Definition [14]. The function $\varphi:[0,\infty)\to[0,\infty)$, is called an altering distance function if the following properties are satisfied:

(1) φ is continuous and non-decreasing.

(2) $\varphi(t)=0$ if and only if t=0.

II. Main Results

The goal of this paper is to present some fixed point theorems for a mapping meeting a cyclical generalised contractive condition in partial 2-metric spaces based on a pair of altering distance

Definition: Let (X,ρ) be a partial 2-metric space and A and B be nonempty closed subsets of X . A mapping T:X→X is called a cyclic (ψ,φ,A,B)-contraction if

(1) ψ and φ are altering distance functions,

(2) A∪B has a cyclic representation w.r.t. T ; that is, T(A)⊆B and T(B)⊆A, and

(3) $\psi(\rho(Tx,Ty,z))\le \psi(max\{\rho(x,y,z),\rho(x,Tx,z),\rho(y,Ty,z)\}) -\varphi(max\{\rho(x,y,z),\rho(y,Ty,z)\})$, for all x∈A, y∈B and z∈X.

Now we introduce our main result.

Theorem : Assume (X,ρ) be a complete partial 2-metric space, let A and B be nonempty closed subsets of X. If T:X→X is a cyclic (ψ,φ,A,B)-contraction, then T has a unique fixed point u∈A∩B.

Proof

Suppose that $x_0 \in A$. Since T:X→X is a cyclic (ψ,φ,A,B)-contraction then T(A)⊆B, if we choose x_1∈B such that $Tx_0=x_1$. Also, if T(B)⊆A, we set x_2∈A such that $Tx_1=x_2$. By continuing, we can construct sequences $\{ x_n \}$ in X such that $Tx_{2n} = x_{2n+1}$ *and* $Tx_{2n+1} = x_{2n+2}$ for $x_{2n} \in A$ and $x_{2n+1} \in B$. Then there exist two cases :

Case (1) If $x_{2n_0+1} = x_{2n_0+2}$ for some $n \in \mathrm{N}$, then $x_{2n_0+1} = Tx_{2n_0} = x_{2n_0+2} = Tx_{2n_0+1}$. Thus, x_{2n_0+1} is a fixed point of T in A∩B.

Case (2) We suppose that $x_{2n+1} \neq x_{2n+2}$ for all $n \in \mathrm{N}$. There are two probabilities of n:

(i) If $n \in \mathrm{N}$ is even, then n=2t for some t∈N. Since

$$\psi\left(\rho\left(Tx,Ty,z\right)\right)\leq \psi\left(max\{\rho\left(x,y,z\right),\rho\left(x,Tx,z\right),\rho\left(y,Ty,z\right)\}\right)-\varphi\left(max\{\rho\left(x,y,z\right),\rho\left(y,Ty,z\right)\}\right). \tag{1}$$

Now we have,

$$\begin{aligned}\psi\left(\rho\left(x_{n+1},x_{n+2},z\right)\right)= {}& \psi\left(\rho\left(x_{2t+1},x_{2t+2},z\right)\right)=\psi\left(\rho\left(Tx_{2t},Tx_{2t+1},z\right)\right)\\ &\leq \psi\left(max\{\rho\left(x_{2t},x_{2t+1},z\right),\rho\left(x_{2t},Tx_{2t},z\right),\rho\left(x_{2t+1},Tx_{2t+1},z\right)\}\right)\\ &\quad-\varphi\left(max\{\rho\left(x_{2t},x_{2t+1},z\right),\rho\left(x_{2t+1},Tx_{2t+1},z\right)\}\right)\\ &=\psi\left(max\{\rho\left(x_{2t},x_{2t+1},z\right),\rho\left(x_{2t},x_{2t+1},z\right),\rho\left(x_{2t+1},x_{2t+2},z\right)\}\right)\\ &\quad-\varphi\left(max\{\rho\left(x_{2t},x_{2t+1},z\right),\rho\left(x_{2t+1},x_{2t+2},z\right)\}\right).\end{aligned}$$

So,

$$\begin{aligned}\psi\left(\rho\left(x_{2t+1},x_{2t+2},z\right)\right)\leq {}& \psi\left(max\{\rho\left(x_{2t},x_{2t+1},z\right),\rho\left(x_{2t+1},x_{2t+2},z\right)\}\right)\\ &-\varphi\left(max\{\rho\left(x_{2t},x_{2t+1},z\right),\rho\left(x_{2t+1},x_{2t+2},z\right)\}\right).\end{aligned} \tag{2}$$

If we choose $max\left\{\rho\left(x_{2t},x_{2t+1},z\right),\rho\left(x_{2t+1},x_{2t+2},z\right)\right\}=\rho\left(x_{2t+1},x_{2t+2},z\right)$.

Then eq.(2) becomes

$$\psi\left(\rho\left(x_{2t+1},x_{2t+2},z\right)\right)\leq\psi\left(\rho\left(x_{2t+1},x_{2t+2},z\right)\right)-\varphi(\rho\left(x_{2t+1},x_{2t+2},z\right).$$

Then $\varphi\left(\rho\left(x_{2t+1},x_{2t+2},z\right)\right)= 0$, and since φ is altering distance function, hence $\rho\left(x_{2t+1},x_{2t+2},z\right)= 0$. From the condition (P2M-1), we get $x_{2t+1} = x_{2t+2}$ this is a contradiction for assumption. Therefore,

$$max\left\{\rho\left(x_{2t},x_{2t+1},z\right),\rho\left(x_{2t+1},x_{2t+2},z\right)\right\}=\rho\left(x_{2t},x_{2t+1},z\right).$$

Then

$$\begin{aligned}\psi\left(\rho\left(x_{2t+1},x_{2t+2},z\right)\right)= {}& \psi\left(\rho\left(x_{n+1},x_{n+2},z\right)\right)\\ &\leq \psi\left(\rho\left(x_{2t},x_{2t+1},z\right)\right)-\varphi(\rho\left(x_{2t},x_{2t+1},z\right)\\ &= \psi\left(\rho\left(x_{n},x_{n+1},z\right)\right)-\varphi(\rho\left(x_{n},x_{n+1},z\right).\end{aligned} \tag{3}$$

(ii) If n∈N is odd, then n=2t+1 for some t∈N. From eq. (1) we have

$$\begin{aligned}\psi\left(\rho\left(x_{n+1},x_{n+2},z\right)\right)= {}& \psi\left(\rho\left(x_{2t+2},x_{2t+3},z\right)\right)=\psi\left(\rho\left(Tx_{2t+1},Tx_{2t+2},z\right)\right)\\ &\leq \psi\left(max\{\rho\left(x_{2t+1},x_{2t+2},z\right),\rho\left(x_{2t+1},Tx_{2t+1},z\right),\rho\left(x_{2t+2},Tx_{2t+2},z\right)\}\right)\\ &-\varphi\left(max\{\rho\left(x_{2t+1},x_{2t+2},z\right),\rho\left(x_{2t+2},Tx_{2t+2},z\right)\}\right)\end{aligned}$$

$$=\psi\left(max\{\rho\left(x_{2t+1},x_{2t+2},z\right),\rho\left(x_{2t+1},Tx_{2t+1},z\right),\rho\left(x_{2t+2},Tx_{2t+2},z\right)\}\right)$$
$$-\varphi\left(max\{\rho\left(x_{2t+1},x_{2t+2},z\right),\rho\left(x_{2t+2},Tx_{2t+2},z\right)\}\right)$$

that is,

$$\psi\left(\rho\left(x_{2t+2},x_{2t+3},z\right)\right)\leq \psi\left(max\{\rho\left(x_{2t+1},x_{2t+2},z\right),\rho\left(x_{2t+2},x_{2t+3},z\right)\}\right)$$
$$-\varphi\left(max\{\rho\left(x_{2t+1},x_{2t+2},z\right),\rho\left(x_{2t+2},x_{2t+3},z\right)\}\right). \quad (4)$$

If we choose

$$max\left\{\rho\left(x_{2t+1},x_{2t+2},z\right),\rho\left(x_{2t+2},x_{2t+3},z\right)\right\}=\rho\left(x_{2t+2},x_{2t+3},z\right).$$

Then eq.(4) becomes

$$\psi\left(\rho\left(x_{2t+2},x_{2t+3},z\right)\right)\leq\psi\left(\rho\left(x_{2t+2},x_{2t+3},z\right)\right)-\varphi\left(\rho\left(x_{2t+2},x_{2t+3},z\right)\right).$$

Then $\varphi\left(\rho\left(x_{2t+2},x_{2t+3},z\right)\right)=0$, hence $\rho\left(x_{2t+2},x_{2t+3},z\right)=0$. From the condition (P2M-1), we get $x_{2t+2}=x_{2t+3}$ this is a contradiction for assumption. Therefore,

$$max\left\{\rho\left(x_{2t+1},x_{2t+2},z\right),\rho\left(x_{2t+2},x_{2t+3},z\right)\right\}=\rho\left(x_{2t+1},x_{2t+2},z\right).$$

Then

$$\psi\left(\rho\left(x_{n+1},x_{n+2},z\right)\right)= \psi\left(\rho\left(x_{2t+2},x_{2t+3},z\right)\right)$$
$$\leq \psi\left(\rho\left(x_{2t+1},x_{2t+2},z\right)\right)-\varphi\left(\rho\left(x_{2t+1},x_{2t+2},z\right)\right)$$
$$= \psi\left(\rho\left(x_{n},x_{n+1},z\right)\right)-\varphi\left(\rho\left(x_{n},x_{n+1},z\right)\right). \quad (5)$$

Eqs. (3) and (5) are the same results, so,

$$\psi\left(\rho\left(x_{n+1},x_{n+2},z\right)\right)\leq\psi\left(\rho\left(x_{n},x_{n+1},z\right)\right)-\varphi\left(\rho\left(x_{n},x_{n+1},z\right)\right)\quad \forall n\in N \quad (6)$$

then we have $\{\rho(x_{n+1},x_{n+2},z):n\in N\}$ is a non-increasing sequence and consequently, there is r≥0 such that the limit

$$lim_{n\to\infty}\rho\left(x_{n+1},x_{n+2},z\right)=r. \quad (7)$$

From eq. (6), by taking as n→∞ and using eq. (7), since ψ and φ are continuous functions, we have

$$\psi\left(r\right)\leq\psi\left(r\right)-\varphi\left(r\right).$$

Therefore, φ(r)=0 if and only if r=0. Then

$$lim_{n\to\infty}\rho\left(x_{n+1},x_{n+2},z\right)=0. \quad (8)$$

From (P2M-2), we get $\rho\left(x_n,x_n,x_n\right)\leq\rho\left(x_n,x_{n+1},z\right)$, by taking the limit as n→∞ and using eq. (8) we get

$$lim_{n\to\infty}\rho\left(x_n,x_n,x_n\right)=0. \quad (9)$$

Since

$$d_{\rho}\left(x_n,x_{n+1},z\right)=3\rho\left(x_n,x_{n+1},z\right)-\rho\left(x_n,x_n,x_n\right)-\rho\left(x_{n+1},x_{n+1},x_{n+1}\right)-\rho\left(z,z,z\right),$$

then $lim_{n\to\infty}d_{\rho}\left(x_n,x_{n+1},z\right)\leq 3lim_{n\to\infty}\rho\left(x_n,x_{n+1},z\right)$,

using eq. (8) in the above equation we get

$$lim_{n\to\infty}d_{\rho}\left(x_n,x_{n+1},z\right)=0 \quad (10)$$

Now we can demonstrate this $\{x_n\}$ is a Cauchy sequence in the metric space $(A\cup B,d_{\rho})$. Consider that $\{x_n\}$ is not a Cauchy sequence in $(A\cup B,d_{\rho})$. Then there exists ε>0 for which we have two subsequences $\{x_{n(i)}\}$ and $\{x_{m(i)}\}$ of $\{x_n\}$ such that $n(i)>m(i)<i$, then

$$d_\rho\left(x_{m(i)},x_{n(i)},z\right)\geq \varepsilon.$$

This means that $d_\rho\left(x_{m(i)},x_{n(i)-1},z\right)<\varepsilon.$ (11)

From (P2M-4), we get

$$\varepsilon \leq d_\rho\left(x_{m(i)},x_{n(i)},z\right)$$

$$\leq d_\rho\left(x_{m(i)},x_{n(i)},x_{n(i)-1}\right)+d_\rho\left(x_{m(i)},x_{n(i)-1},z\right)+d_\rho\left(x_{n(i)-1},x_{n(i)},z\right)-d_\rho\left(x_{n(i)-1},x_{n(i)-1},x_{n(i)-1}\right)$$

$$\leq d_\rho\left(x_{m(i)},x_{n(i)},x_{n(i)-1}\right)+d_\rho\left(x_{m(i)},x_{n(i)-1},z\right)+d_\rho\left(x_{n(i)-1},x_{n(i)},z\right)$$

Taking $i\to\infty$ we get

$$\varepsilon \leq lim_{i\to\infty} d_\rho\left(x_{m(i)},x_{n(i)},z\right)$$

$$\leq lim_{i\to\infty} d_\rho\left(x_{m(i)},x_{n(i)},x_{n(i)-1}\right)+lim_{i\to\infty} d_\rho\left(x_{m(i)},x_{n(i)-1},z\right)$$

$$+lim_{i\to\infty} d_\rho\left(x_{n(i)-1},x_{n(i)},z\right),$$

and using eqs. (8), (10) and (11) in the above equation we get

$$\varepsilon \leq lim_{i\to\infty} d_\rho\left(x_{m(i)},x_{n(i)},z\right)\leq 0+\varepsilon+0=\varepsilon.$$

We have, $\varepsilon \leq lim_{i\to\infty} d_\rho\left(x_{m(i)},x_{n(i)},z\right)\leq\varepsilon$.

Then $lim_{i\to\infty} d_\rho\left(x_{m(i)},x_{n(i)},z\right)=\varepsilon$.

$$d_\rho\left(x_{m(i)},x_{n(i)},z\right)= 3\rho\left(x_{m(i)},x_{n(i)},z\right)-\rho\left(x_{m(i)},x_{m(i)},x_{m(i)}\right)$$

Since $-\rho\left(x_{n(i)},x_{n(i)},x_{n(i)}\right)-\rho\left(z,z,z\right).$.

Letting $i\to\infty$ in the above equation and using eq. (9), then

$$lim_{i\to\infty} d_\rho\left(x_{m(i)},x_{n(i)},z\right)= \varepsilon = 3lim_{i\to\infty}\rho\left(x_{m(i)},x_{n(i)},z\right)-lim_{i\to\infty}\rho\left(x_{m(i)},x_{m(i)},x_{m(i)}\right)$$

$$-lim_{i\to\infty}\rho\left(x_{n(i)},x_{n(i)},x_{n(i)}\right)-lim_{i\to\infty}\rho\left(z,z,z\right)$$

$$= 3lim_{i\to\infty}\rho\left(x_{m(i)},x_{n(i)},z\right).$$

Then $lim_{i\to\infty}\rho\left(x_{m(i)},x_{n(i)},z\right)=\frac{\varepsilon}{3}$ (12)

From eq. (1) we have

$$\psi(\rho(x_{m(i)},x_{n(i)},z)) = \psi(\rho(Tx_{m(i)},Tx_{n(i)},z))$$

$$\leq \psi(\max\{\rho(x_{m(i)-1},x_{n(i)-1},z),\rho(x_{m(i)-1},Tx_{m(i)-1},z),$$

$$\rho(x_{n(i)-1},Tx_{n(i)-1},z)\})-\phi(\max\{\rho(x_{m(i)-1},x_{n(i)-1},z),$$

$$\rho(x_{n(i)-1},Tx_{n(i)-1},z)\})$$

$$= \psi(max\{\rho(x_{m(i)-1},x_{n(i)-1},z),\rho(x_{m(i)-1},x_{m(i)-1},z),$$
$$\rho(x_{n(i)-1},x_{n(i)-1},z)\}) - \varphi(max\{\rho(x_{m(i)-1},x_{n(i)-1},z),$$
$$\rho(x_{n(i)-1},x_{n(i)-1},z)\}).$$

Letting i→∞ and from eq. (12), which the continuity of ψ and φ, we get that

$$\psi\left(\frac{\varepsilon}{3}\right) \le \psi\left(\frac{\varepsilon}{3}\right) - \varphi\left(\frac{\varepsilon}{3}\right).$$

Therefore, $\varphi\left(\frac{\varepsilon}{3}\right)=0$, since φ is alternig distance function, hence ε=0, which is a contradiction.

Thus the sequence $\{x_n\}$ is a Cauchy sequence in the metric space $(A\cup B, d_\rho)$. If (X,ρ) is complete and A∪B is a closed subspace of (X,ρ), using Lemma , then (A∪B,ρ) is also complete.

Then the sequence $\{x_n\}$converges in the metric space $(A\cup B, d_\rho)$, say

$$lim_{i\to\infty} d_\rho(x_n,u,z)=0,$$

once more from Lemma , we find

$$lim_{n\to\infty}\rho(x_n,u,z) = lim_{n,m\to\infty}\rho(x_n,x_m,z) = \rho(u,u,u) \Leftrightarrow lim_{n\to\infty} d_\rho(x_n,u,z)=0. \quad (13)$$

Since the sequence $\{x_n\}$ is a Cauchy sequence in the metric space $(A\cup B, d_\rho)$, we have

$$lim_{n,m\to\infty} d_\rho(x_n,x_m,z)=0. \quad (14)$$

Since $d_\rho(x_n,x_m,z) = 3\rho(x_n,x_m,z) - \rho(x_n,x_n,x_n) - \rho(x_m,x_m,x_m) - \rho(z,z,z),$

letting n,m→∞ in the above equation and from eqs. (9) and (14), we obtain

$$lim_{n\to\infty}\rho(x_n,x_m,z)=0. \quad (15)$$

From eq. (15) into eq. (13) we get

$$lim_{n\to\infty}\rho(x_n,u,z) = \rho(u,u,u) = 0 \quad (16)$$

Then the sequence $\{x_n\}$ is a sequence in A, and since A is closed in (X,ρ), we have u∈A. Similarly, we have get u∈B, that is $u\in A\cap B$. From (P2M-4), we get

$$\rho(x_n,Tu,z) \le \rho(x_n,Tu,u) + \rho(x_n,u,z) + \rho(u,Tu,z) - \rho(u,u,u).$$

Letting n→∞ in the above inequalities and using eqs. (9) and (15) we have

$$lim_{n\to\infty}\rho(x_n,Tu,z) \le lim_{n\to\infty}\rho(x_n,Tu,u) + lim_{n\to\infty}\rho(u,Tu,z).$$

Then $lim_{n\to\infty}\rho(x_n,Tu,z) - lim_{n\to\infty}\rho(u,Tu,z) \le lim_{n\to\infty}\rho(x_n,Tu,u).$ (17)

Also,

$$\rho(u,Tu,z) \le \rho(u,Tu,x_n) + \rho(u,x_n,z) + \rho(x_n,Tu,z) - \rho(x_n,x_n,x_n).$$

Letting n→∞ in the above inequalities and using eqs. (9) and (16) we have

$$lim_{n\to\infty}\rho(u,Tu,z) - lim_{n\to\infty}\rho(x_n,Tu,z) \le lim_{n\to\infty}\rho(u,Tu,x_n). \quad (18)$$

From eqs. (17) and (18) we get

$$lim_{n\to\infty}\rho(x_n,Tu,z) = \rho(u,Tu,z). \quad (19)$$

Now, we prove that Tu=u. Since $\{x_n\}\in A$ and u∈B, from eq. (1) we have

$$\psi\left(\rho\left(x_{n+1},Tu,z\right)\right)= \psi\left(\rho\left(Tx_n,Tu,z\right)\right)$$
$$\leq \psi\left(max\{\rho\left(x_n,u,z\right),\rho\left(u,Tu,z\right),\rho\left(x_n,Tx_n,z\right)\}\right)$$
$$-\varphi\left(max\{\rho\left(x_n,u,z\right),\rho\left(u,Tu,z\right)\}\right).$$

So,

$$\psi\left(\rho\left(x_{n+1},Tu,z\right)\right)\leq \psi\left(max\{\rho\left(x_n,u,z\right),\rho\left(u,Tu,z\right),\rho\left(x_n,x_{n+1},z\right)\}\right)$$
$$-\varphi\left(max\{\rho\left(x_n,u,z\right),\rho\left(u,Tu,z\right)\}\right).$$

Letting n→∞ in the above inequalities and using eqs. (15) , (16) we get,

$$\psi\left(lim_{n\to\infty}\rho\left(x_{n+1},Tu,z\right)\right)\leq\psi\left(\rho\left(u,Tu,z\right)\right)-\varphi\left(\rho\left(u,Tu,z\right)\right).$$

Using eq. (19) we get

$$\psi\left(\rho\left(\mathrm{u},Tu,z\right)\right)\leq\psi\left(\rho\left(u,Tu,z\right)\right)-\varphi\left(\rho\left(u,Tu,z\right)\right).$$

Then we get $\varphi\left(\rho\left(u,Tu,z\right)\right)=0$ and since φ is altering distance function, hence $\rho\left(u,Tu,z\right)=0$. From the condition (P2M-1), we get Tu=u. Therefore u is a fixed point of T . We show that the fixed point is unique, we set v be any another fixed of T in $A\cap B$ such that $v=Tv,\ u\neq v$. Since and $v\in A\cap B\subseteq B$,
we find

$$\psi\left(\rho\left(u,v,z\right)\right)= \psi(\rho\left(Tu,Tv,z\right)$$
$$\leq \psi(max\left\{\rho\left(u,v,z\right),\rho\left(u,Tu,z\right),\rho\left(v,Tv,z\right)\right\}$$
$$-\varphi\left(max\{\rho\left(u,v,z\right),\rho\left(u,Tu,z\right)\}\right)$$
$$= \psi\left(\rho\left(u,v,z\right)\right)-\varphi\left(\rho\left(u,v,z\right)\right).$$

Thus $\varphi\left(\rho\left(u,\mathrm{v},z\right)\right)=0$ and where the function φ is an altering distance function, hence $\rho\left(u,v,z\right)=0$. Therefore, $u=v$. Hence the proof.

Example: Let $X=\left\{0,1\right\}$, and establish the P2M space on X by

$\rho\left(\mathrm{x},\mathrm{y},\mathrm{z}\right)=\begin{cases}2 & \text{if } x=y=z=0\\ 1 & \text{otherwise}\end{cases}$. Also, define the mapping $T:X\to X$ by $T\left(x\right)=\dfrac{x^2}{1+x}$ and the functions $\psi,\varphi:\left[0,\infty\right)\to\left[0,\infty\right)$ by $\psi\left(t\right)=2t$ and $\varphi\left(t\right)=0$. Take $A=\left[0,\frac{1}{2}\right]$ and $B=\left[0,1\right]$.

This example satisfy all the hypothesis of the above theorem.

Proof. Since $A=\left[0,\frac{1}{2}\right]$ and $T\left(x\right)=\dfrac{x^2}{1+x}$, we get $T\left(0\right)=0,\quad T\left(\frac{1}{2}\right)=\frac{1}{6}$, then $T\left(A\right)=\left[0,\frac{1}{6}\right]$. Note that $T\left(A\right)\subset B.$ From $B=\left[0,1\right]$,

we get $T\left(0\right)=0,\ T\left(1\right)=\frac{1}{2}$, then $T\left(B\right)=\left[0,\frac{1}{2}\right]$.

Note that $T\left(B\right)\subseteq A.$ Therefore $A\cup B$ has a cyclic representation of T .

Set $x\in A$ and $y\in B$, without sacrificing generality, We can presume that $x\leq y$.

We prove that inequality

$$\psi\left(\rho\left(Tx,Ty,z\right)\right)\leq \psi\left(max\{\rho\left(x,y,z\right),\rho\left(x,Tx,z\right),\rho\left(y,Ty,z\right)\}\right)$$
$$-\varphi\left(max\{\rho\left(x,y,z\right),\rho\left(y,Ty,z\right)\}\right).$$

Since $\psi\left(\rho\left(Tx,Ty,z\right)\right)=\psi\left(\rho\left(\left(\frac{x^2}{1+x}\right),\left(\frac{y^2}{1+y}\right),z\right)\right)=\psi\left(1\right)=2.$

Since

$$\psi\left(max\{\rho\left(x,y,z\right),\rho\left(x,Tx,z\right),\rho\left(y,Ty,z\right)\}\right)-$$
$$\varphi\left(max\{\rho\left(x,y,z\right),\rho\left(y,Ty,z\right)\}\right)$$
$$=\ \psi\left(max\{1,1,1\}\right)-\varphi\left(max\{1,1\}\right)$$
$$=\ \psi\left(1\right)-\varphi\left(1\right)=2-0=2.$$

CONCLUSION

In this paper, introduced the new types of metric spaces and generalized some definitions and concepts in new metric spaces. Finally , we Study a common fixed point theorems for more than one mapping which defined on new metric spaces under general contractive conditions.

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